Self-Paced Study Guide in Trigonometry

MIT INSTITUTE OF TECHNOLOGY

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Contents

1 Credits

The review modules were written by Professor A. P. French (Physics Department) and Adeliada Moranescu (MIT Class of 1994). The problems and solutions were written by Professor Arthur Mattuck (Mathematics Department). This document was originally produced by the Undergraduate Academic Affairs Office, August, 1992, and edited and transcribed to LaTeX by Tea Dorminy (MIT Class of 2013) in August, 2010.

2 How to Use the Self-Paced Review Module

The *Self-Paced Review* consists of review modules with exercises; problems and solutions; self-tests and solutions; and self-evaluations for the four topic areas Algebra, Geometry and Analytic Geometry, Trigonometry, and Exponentials & Logarithms. In addition, previous *Diagnostic Exams* with solutions are included. Each topic area is independent of the others.

The *Review Modules* are designed to introduce the core material for each topic area. A numbering system facilitates easy tracking of subject material. For example, in the topic area Algebra, the subtopic Linear Equations is numbered by 2.3. Problems and the self-evaluations are categorized according to this numbering system.

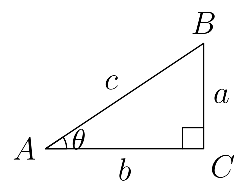
When using the *Self-Paced Review*, it is important to differentiate between concept learning and problem solving. The review modules are oriented toward refreshing concept understanding while the problems and self-tests are designed to develop problem solving ability. When reviewing the modules, exercises are liberally sprinkled throughout the modules and should be solved while working through the module. The problems should be attempted without looking at the solutions. If a problem cannot be solved after at least two honest efforts, then consult the solutions. The tests should be taken only when both an understanding of the material and a problem solving ability have been achieved. The self-evaluation is a useful tool to evaluate one's mastery of the material. The previous Diagnostic Exams should provide the finishing touch.

3 Trigonometry Self-Paced Review Module

As you probably know, trigonometry is just "the measurement of triangles", and that is how it got started, in connection with surveying the earth and the universe. But it has become an essential part of the language of mathematics, physics, and engineering.

3.1 Right Triangles

The simplest place to begin this review is with right triangles. We just have an angle θ ($0° < \theta < 90°$), and the lengths of the sides a, b, c. With this labeling of the sides, we have:



a is the side *opposite* to θ;

b is the side *adjacent* to θ;

c is the *hypotenuse* (literally the "stretched side").

From these we construct the three primary trigonometric functions — sine, cosine, and tangent:

$$\sin\theta = \frac{a}{c}; \qquad \cos\theta = \frac{b}{c}; \qquad \tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}$$

Some people remember these through a mnemonic trick — the nonsense word SOHCAHTOA:

$$\textbf{S}\text{ine} = \frac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}}; \quad \textbf{C}\text{osine} = \frac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}}; \quad \textbf{T}\text{angent} = \frac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}}$$

Perhaps you yourself learned this. But you'll be much better off if you simply *know* these relations as a sort of reflex and don't have to think about which ratio is which.

You will also need to be familiar with the reciprocals of these functions —

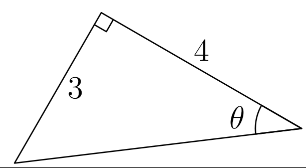
$$\textit{cosecant} = 1/\text{sine}; \qquad \textit{secant} = 1/\text{cosine}; \qquad \textit{cotangent} = 1/\text{tangent}:$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{c}{a}; \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}; \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{b}{a}.$$

If we wish, we can of course express the hypotenuse c in terms of a and b with the help of Pythagoras' Theorem:

$$c^2 = a^2 + b^2, \text{ so } c = \sqrt{a^2 + b^2} = (a^2 + b^2)^{1/2}$$

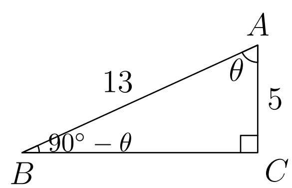
Exercise 3.1.1:
Cover up the formulas above. Then find $\sin\theta, \cos\theta, \tan\theta, \csc\theta, \sec\theta, \cot\theta$ in the triangle shown here. (We've deliberately drawn it in a non-standard orientation; you need to be able to handle that sort of thing.)



Note: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

Exercise 3.1.2:
In this triangle, find:

1. $\sin(90^\circ - \theta)$
2. $\sin\theta + \cos(90^\circ - \theta)$
3. $\tan\theta + \cot(90^\circ - \theta)$
4. $\sec\theta + \csc(90^\circ - \theta)$



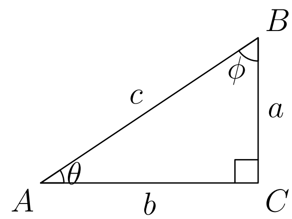
[Note: $90^\circ - \theta$ is a perfectly valid name for the angle at B, though for some purposes we might want to call it, say, β for simplicity. But the important thing here is just to get the relation of the sines and cosines, etc., straight. Here, θ is what we might call the *primary* angle, $90^\circ - \theta$ is the *co-angle* (complementary angle). The above exercise is designed to make the point that the sine, tangent and secant of the angle θ have the same values as the co-sine, co-tangent, and co-secant of the co-angle $(90^\circ - \theta)$ — and vice versa.]

Exercise 3.1.3:
Sorry, folks. No picture this time. *You* draw the triangle.
If A is an acute angle and $\sin A = 7/25$, find all the other trigonometric functions of the angle A.

IMPORTANT!

It's not enough to know the *definitions* of the various trigonometric functions. You also need to be able to *use* them to find the length of any side of a right triangle in terms of any other side and one of the angles. That is, in the triangle ABC, in which C is the right angle, you should be familiar with the following relationships:

$$a = b\tan\theta = c\sin\theta = c\cos\phi = b\cot\phi$$
$$b = a\cot\theta = c\cos\theta = c\sin\phi = a\tan\phi$$
$$c = a\csc\theta = b\sec\theta = a\sec\phi = b\csc\phi$$



In particular, the relations $a = c\sin\theta$, and $b = c\cos\theta$, correspond to the process of breaking up a linear displacement or a force into two perpendicular components. This operation is one that you will need to perform over and over again in physics problems. Learn it now so that you have it ready for instant use later.

Exercise 3.1.4:
In a right triangle labeled as above, find:

a) BC, if $A = 20°$, $AB = 5$;

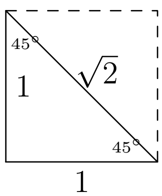
b) AC, if $B = 40°$, $BC = 8$;

c) AB, if $A = 53°$, $AC = 6$

3.2 Some Special Triangles

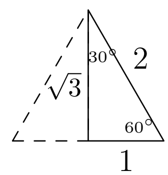
We've already used some special triangles in Section 1, above — right triangles in which all three sides can be expressed as integers: (3, 4, 5), (5, 12, 13), and (7, 24, 25). It's very convenient to be familiar with such triangles, for which Pythagoras' Theorem becomes just a relation between the squares of the natural numbers. (You might like to amuse yourself hunting for more examples.) And people who set examinations are fond of using such triangles, to save work for the students who write the exams and for the people who grade them. So there can be a very practical advantage in knowing them!

And there are some other special triangles that you should know inside out. Look at the two triangles below. No doubt you're familiar with both of them. The first is half of a square and the second is half of an equilateral triangle. Their angles and principal trigonometric functions are as shown.



$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \cos 45^\circ$$

$$\tan 45^\circ = 1$$



$$\sin 30^\circ = \tfrac{1}{2} = \cos 60^\circ$$

$$\cos 30^\circ = \tfrac{\sqrt{3}}{2} = \sin 60^\circ$$

$$\tan 30^\circ = \tfrac{1}{\sqrt{3}} = \cot 60^\circ$$

These triangles, too, often show up in quizzes and examinations. You will very likely be expected to know them in tests where calculators are not allowed. More importantly, though, they should become part of your stock-in-trade of known numerical values that you can use for problem-solving. Getting an *approximate* answer to a problem can often be very useful. You might, for example, be doing a problem in which the cosine of 58.8° shows up. Maybe you don't have your calculator with you – or maybe it has decided to break down. If you know that $\cos 60^\circ = 1/2$, you can use this as a good approximation to the value you really want. Also, since (for angles between 0° and 90°) the cosine gets smaller as the angle gets bigger, you will know that the true value of $\cos 58^\circ$ is a little *bigger* than 1/2; that can be very useful information too.

Exercise 3.2.1:
Take a separate sheet of paper, draw the 45°/45°/90° and 30°/60°/90° triangles, and write down the values of all the trig functions you have learned. *Convert the values to decimal form also.*

Exercise 3.2.2:
Given: this triangle, with $C = 90°$. Find all angles and lengths of the sides if:

a) $A = 30°$,$b = 6$

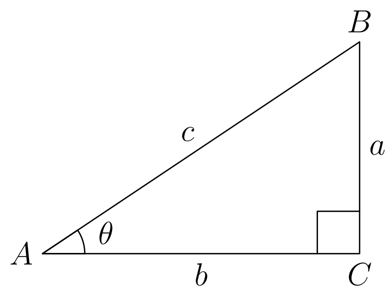
b) $a = 13, b = 13$

c) $B = 30°$, $c = 10$

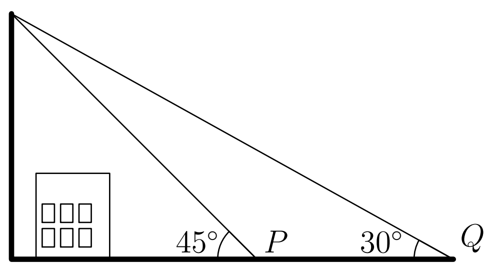
d) $A = 45°$, $c = 12$

e) $c = 1, b = \sqrt{2}$

f) $c = 4, b = 4$

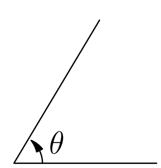


Exercise 3.2.3:
A tall flagpole stands behind a building. Standing at point P, you observe that you must look up at an angle of 45° to see the top of the pole. You then walk away from the building through a distance of 10 meters to point Q. From here, the line to the top of the pole makes an angle of 30° to the horizontal. How high is the top of the pole *above eye level*?
[If you don't see how to approach this problem, take a peek at the solution — just the diagram at first.]



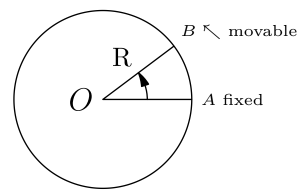
3.3 Radian Angle Measure

A given angle θ is uniquely defined by the intersection of two straight lines. But the actual *measure* of the angle can be expressed in different ways. For most practical purposes (e.g., navigation) we use the division of the full circle into 360 degrees, and measure angles in terms of degrees, minutes, and seconds of arc. But in mathematics and physics, angles



are usually expressed in *radians*. Radians are much more useful than degrees when you are studying functions, graphs, and such things as periodic motion. This is because radians simplify all calculus formulas for trig functions. The price we pay for simplicity is that we need to introduce the fundamental constant π. But that is worth understanding anyway.

Imagine a circle, with an angle formed between two radii. Suppose that one of the radii (the horizontal one) is fixed, and that the other one is free to rotate about the center so as to define any size of angle we please. To make things simple, choose the circle to have radius $R = 1$ (inch, centimeter, whatever — it doesn't matter). We can take some flexible material (string or thread) and cut off a unit length of it (the same units as we've used for R itself). We place one end of the string at the end A of the fixed radius, and fit the string around the contour of the circle. If we now put the end B of the movable radius at the other end of the string, the angle between the two radii is *1 radian* (Mnemonic: 1 radian = 1 radius-angle).

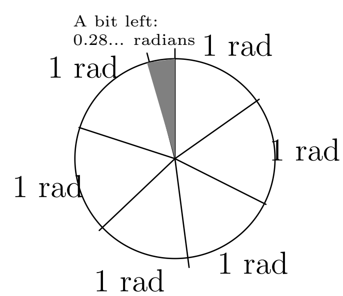


To put it formally:

> A radian is the measure of the angle that cuts off an arc of length 1 on the unit circle.

Another way of saying this is that an arc of unit length *subtends* an angle of 1 radian at the center of the unit circle.

[If you don't like unit circles, you can say that *an angle of one radian cuts off an arc equal in length to the radius on a circle of any given radius*]



If we take a length of string equal to the radius and go all the way around the circle, we find that we can fit in 6 lengths plus a bit more (about 0.28 of the radius)[1]. For one-half of the circle, the arc length is equal (to an accuracy of two decimal places) to

[1]Note that, if we had use the string to step out straight chords instead of arcs, we would complete the circle with *exactly* 6 lengths, forming a regular hexagon. You can thus guess that one radian is a bit less than 60°.

3.14 radii. Surprise: $3.14 = \pi$! (No, *not* a surprise. This is how π is defined.) So we say there are π radians in 180°, and we have:

- 2π radians is equivalent to a 360° rotation
- π radians is equivalent to a 180° rotation

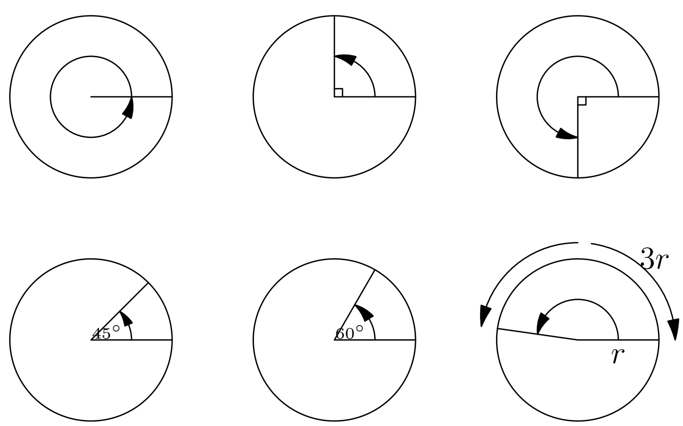
The abbreviation rad is often used when we write the value of an angle in radians.

Numerically, we have:

$$1 \text{ rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ.$$

Now it's your turn:

Exercise 3.3.1:
Evaluate the following angles in radians:



Remember, θ in radians is $\frac{6.0\text{pt}\widehat{AB}}{r}$, or $6.0\text{pt}\widehat{AB} = r\theta$ (with θ in rad).

Thus, for example, on a circle with radius 10 cm, an angle of n radians cuts off (intercepts) an arc of length $10n$ cm.

Exercise 3.3.2:
Express in radian measure:

a) 30 degrees
b) 135 degrees
c) 210 degrees
d) 315 degrees
e) 160 degrees
f) 10 degrees

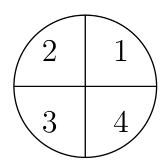
Exercise 3.3.3:
Express in degree measure:

a) $\frac{\pi}{3}$ rad
b) $\frac{5}{9}\pi$ rad
c) $\frac{\pi}{24}$ rad
d) $\frac{\pi}{4}$ rad
e) $\frac{7}{6}\pi$ rad

How about some trig functions of angles expressed in radian measure?

Exercise 3.3.4:
Evaluate:
a) $\sin\frac{\pi}{6}$
b) $\cos\frac{\pi}{4}$.
c) Is it true that $\sin\frac{\pi}{18} = \cos\frac{4\pi}{9}$?

You should be familiar with the *signs* of the trigonometric functions for different ranges of angle. This is traditionally done in terms of the four *quadrants* of the complete circle. It is convenient to make a table showing the signs of the trig functions without regard to their actual values:



	1st Quad	2nd Quad	3rd Quad	4th Quad
sin(csc)	+	+	-	-
cos(sec)	+	-	-	+
tan(cot)	+	-	+	-

This means that there are always *two* angles between 0 and 2π that have the same sign and magnitude of any given trig function.

The mnemonic "**A**ll **S**tudents **T**ake **C**alculus", while not a true statement (except at MIT and Caltech and maybe a few other places) can remind you which of the main trig functions are *positive* in the successive quadrants.

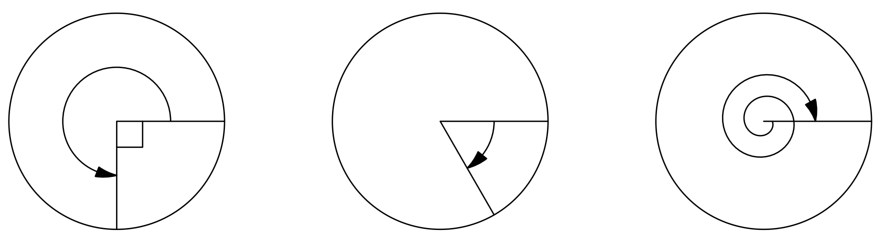
Exercise 3.3.5:
Fill in the following table, giving both signs and magnitudes of the trig functions:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$\pi/4$				$5\pi/6$			
$\pi/3$				$7\pi/6$			
$2\pi/3$				$3\pi/2$			
$3\pi/4$				$7\pi/4$			

If we are just dealing with triangles, all of the angles are counted positive and are less than 180° (i.e. $< \pi$). But angles less than 0 or more than π are important for describing rotations. For this purpose we use a definite sign convention:

> Counterclockwise rotations are positive; clockwise rotations are negative

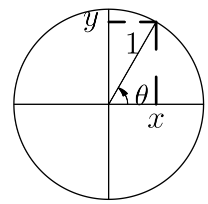
Here are some examples:



$\frac{3\pi}{2}$ rad $-\frac{\pi}{3}$ rad -4π rad

When we are dealing with rotations, we often talk in terms of numbers of *revolutions*. Since 1 revolution (rev) is equivalent to 2π rad, it is easy to convert from one to the other: just *multiply revs by* 2π *to get rad*, or *divide rads by* 2π *to get revs*.

3.4 Trigonometric Functions as Functions



We have emphasized the usefulness of knowing the values of sin, cos, tan, etc. of various specific angles, but of course the trig functions really *are* functions of a continuous variable — the angle θ that can have any value between $-\infty$ and ∞. It is important to have a sense of the appearance and properties of the graphs of these functions.

Drawing the unit circle can be of help in visualizing these functions. If we draw a unit circle, and give it x and y axes as shown, then we have:

$$\sin\theta = \frac{y}{1} = y; \cos\theta = \frac{x}{1} = x; \tan\theta = \frac{y}{x}$$

Notice the following features:

1. Every time we go through an integral multiple of 2π, we come back to the same point on the unit circle. This means that sin, cos, and tan are periodic functions of θ with period 2π; i.e.

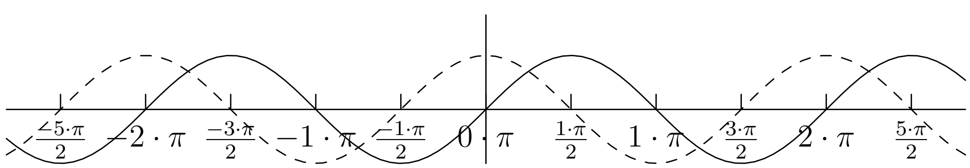
$$\sin(\theta + 2\pi) = \sin\theta, \text{ and similarly for } \cos \text{ and } \tan.$$

2. The values of sin and cos never go outside the range between $+1$ and -1; they oscillate between these limits.

3. $\cos\theta$ is $+1$ at $\theta = 0$; $\sin\theta = +1$ at $\theta = \pi/2$. More generally, one can put:

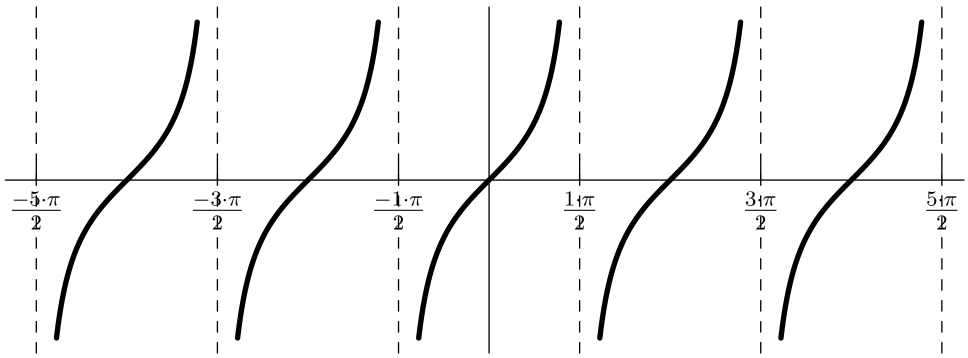
$$\cos\theta = \sin(\theta + \pi/2) \text{ or } \sin\theta = \cos(\theta - \pi/2).$$

Thus the whole cosine curve is shifted through $\pi/2$ negatively along the θ axis relative to the sine curve, as shown below:



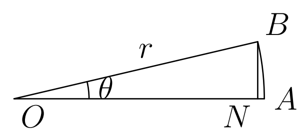
It is easy to construct quite good sketches of these functions with the help of the particular values of sin and cos with which you are familiar.

4. The tangent function becomes infinitely large (+ or -) at odd multiples of $\pi/2$, where the value of x on the unit circle goes to 0. It is made of infinitely many separate pieces, as indicated below:



Like the sine function, the tangent function is zero whenever θ is an integral multiple of π. Moreover, $\tan\theta$ is almost equal to $\sin\theta$ if θ is a *small* angle (say less than about 10° or .2 rad). For such small angles, both sin and tan are also almost equal to θ itself as measured in radians. This can be seen from the diagram here. We have the following relationships:

$$\begin{aligned}
\theta &= \frac{AB}{r} = \frac{AB}{OA} \geq \frac{BN}{OA}; \\
\sin\theta &= \frac{BN}{r} = \frac{BN}{OA}; \\
\tan\theta &= \frac{BN}{ON} \geq \frac{BN}{OA}.
\end{aligned}$$



Thus,

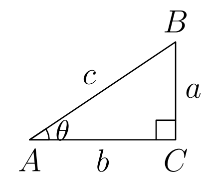
$$\sin\theta \approx \theta \approx \tan\theta$$

These results are the basis of many useful approximations.

Exercise 3.4.1:
Using your calculator, find $\sin\theta$ and $\tan\theta$ for $\theta = 0.1$ rad, 0.15 rad, 0.2 rad, 0.25 rad. (If you do this, you will be able to see that it is always true that $\sin\theta \approx \theta \approx \tan\theta$ for these small angles.)

3.5 Trigonometric Identities

You know that the various trig functions are closely related to one another. For example, $\cos\theta = 1/\sec\theta$. Since this is true for *every* θ, this is an *identity*, not an equation that can be solved for θ. Trigonometric identities are very useful for simplifying and manipulating many mathematical expressions. You ought to know a few of them.



1) Probably the most familiar, and also one of the most useful, is the one based on Pythagoras' Theorem and the definitions of sin and cos:

$$c^2 = a^2 + b^2, \text{ with } \sin\theta = a/c, \cos\theta = b/c;$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Dividing through by $\cos^2\theta$ or $\sin^2\theta$ gives two other identities useful for calculus:[2]

$$\tan^2\theta + 1 = \sec^2\theta; \qquad 1 + \cot^2\theta = \csc^2\theta.$$

2) Two other very simple identities are:

$$\sin(-\alpha) = -\sin\alpha; \qquad \cos(-\beta) = \cos\beta.$$

(We say that $\sin\theta$ is an *odd function* of θ and $\cos\theta$ is an *even function* of θ.)

3) Angle addition formulas: Given any two angles α and β,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

If you learn these formulas, you can easily construct the formulas for sin or cos of the *difference* of two angles. (Just use the odd/even properties of sin and cos.)

[2]Note that $(\sin\theta)^2$ is often abbreviated $\sin^2\theta$, and similarly for the other trigonometric functions. Do not confuse this with $\sin(\sin\theta)$, even though for many other functions $f^2(x)$ does mean $f(f(x))$.

Exercise 3.5.1:
Use the angle addition formulas to evaluate the following:

a) $\sin(\theta + \frac{3\pi}{2})$ c) $\sin(\theta + \frac{\pi}{6})$

b) $\cos(\theta - \frac{\pi}{4})$ d) $\cos(\theta + \frac{7\pi}{4})$

4) Even if you don't memorize the general results in (3), you should certainly know the formulas obtained when you put $\alpha = \beta = \theta$ — the *double-angle formulas*:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

By using the second of these in reverse, you can develop half-angle formulas:

Exercise 3.5.2:
Prove the half-angle formulas:

$$\sin\frac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{2}}; \qquad \cos\frac{1}{2}\alpha = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

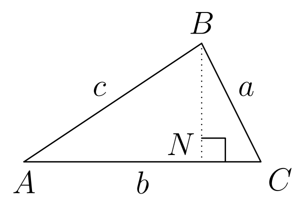
Exercise 3.5.3:
Using the results of Exercise **??**, find $\sin 22.5°$ and $\cos 22.5$.

3.6 Sine and Cosine Laws for the General Triangle

Not everything can be done with right triangles, and you should be familiar with two other sets of identities that apply to a triangle of any shape. Rather than memorizing these forms, you should know how to use them to find angles and side lengths. It's also useful to see how they are derived, namely by dropping a perpendicular and using Pythagoras' Theorem:

3.6.1 The Laws of Sines

For any triangle ABC, labeled as in the diagram:



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This result follows from considering the length of a perpendicular drawn from any angle to the opposite side. Such a perpendicular (BN in our diagram) can be calculated in two ways:

$$BN = c \sin A = a \sin C.$$

Rearranging this gives $\frac{\sin A}{a} = \frac{\sin C}{c}$, and it is pretty obvious that considering either of the other two perpendiculars will complete the relationships.

We can use the law of sines to solve a triangle if we know *one angle and the length of the side opposite to it, plus one other datum* – either another angle or the length of another side. (We can always make use of the fact that the angles of any triangle add up to 180°.)

Exercise 3.6.1:
In a triangle labeled as in the diagram above, let $A = 30°$, $a = 10$, and $C = 135°$. Find B, b, and c.

Exercise 3.6.2:
In another triangle, suppose $B = 50°$, $b = 12$, $c = 15$. Find A, C, and a.

3.6.2 The Law of Cosines

This is useful if we do *not* know the values of an angle and its opposite side. What that means, essentially, is that we are given the value of at most one angle. If this is the angle A in the standard diagram, the Law of Cosines states that:

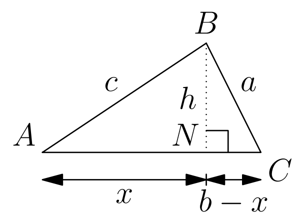
$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Thus, if b, c, and A are given, we can calculate the length of the third side, a.

The Law of Cosines is an extension of the Pythagorean Theorem, and we prove it by using the Pythagorean Theorem. Take a triangle labeled as before, and again draw a perpendicular from angle B onto AC. Let $BN = h$ and let AN be x, so that $NC = b - x$. Then in $\triangle ABN$ we have $c^2 = x^2 + h^2$, and in $\triangle BCN$ we have $a^2 = (b-x)^2 + h^2$.

Combining these gives $c^2 - a^2 = 2bx - b^2$. But $x = c \cos A$; substituting this and rearranging then gives

$$a^2 = b^2 + c^2 - 2bc \cos A$$



as already stated. Doing similar calculations based on drawing perpendiculars from A and C gives similar equations for b^2 and c^2 in terms of c, a, B and a, b, C respectively. (But you don't need to do new calculations: just permute the symbols in the first equation.)

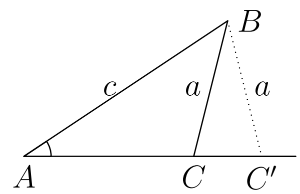
The Law of Cosines also allows us to find the angle A if we are given the lengths of all three sides. For this purpose it can be rewritten as:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

As soon as one angle has been determined in this way, we can use the Law of Sines to do the rest.

Exercise 3.6.3:
In a triangle labeled as above, $a = 5$, $b = 10$, and $C = 135°$. Find c, A, and B.

Exercise 3.6.4:
In another triangle, suppose $a = 10$, $b = 20$, $c = 25$. Find all the angles.
[Draw a reasonably good sketch to see what the triangle looks like, and remember that $\sin \theta$ and $\sin(180° - \theta)$ are equal.]



Warning!: A triangle is completely defined if we know the lengths of two of its sides and the included angle, or any two of its angles and the length of one side. Use of the law of sines or the law of cosines, as appropriate, will give us the rest of the information. However, if we know the lengths of two sides, plus an angle that is *not* the angle between them, there *may* be an ambiguity. The diagram here shows an example of this. If we are given the angle A and the sides a and c, there may be two possible solutions, according to whether the angle C is less than 90° or greater than 90°. The magnitude of $\cos C$ is defined, but its sign is not. This ambiguity will exist whenever a is less than c — i.e., if the length of the side opposite to the given angle is shorter than the adjacent side.

Exercise 3.6.5:
A triangle has $A = 30°$, $a = 6$, $c = 10$. Find B, C, and b.

3.7 Answers to Exercises

Note: In a few cases (Exercises **??**, **??**, **??**, **??**), answers are not given, because you can so easily check them for yourself.

Exercise **??**: $\frac{3}{5}$; $\frac{4}{5}$; $\frac{3}{4}$; $\frac{5}{3}$; $\frac{5}{4}$; $\frac{4}{3}$.

Exercise **??**:

$$\begin{aligned}
\sin(90° - \theta) &= \frac{5}{13};\\
\sin\theta + \cos(90° - \theta) &= \frac{12}{13} + \frac{12}{13} = \frac{24}{13};\\
\tan\theta + \cot(90° - \theta) &= \frac{12}{5} + \frac{12}{5} = \frac{24}{5};\\
\sec\theta + \csc(90° - \theta) &= \frac{13}{5} + \frac{13}{5} = \frac{26}{5}
\end{aligned}$$

Exercise **??**: $\cos A = \frac{24}{25}$; $\tan A = \frac{7}{24}$; $\csc A = \frac{25}{7}$; $\sec A = \frac{25}{24}$; $\cot A = \frac{24}{7}$

Exercise **??**: (a) $5\sin 20° \cong -1.71$; (b) $8\tan 40° \cong 6.71$; (c) $6\sec 53° \cong 9.97 \cong 10$ — a 3:4:5 triangle. (More precisely, the angles in such a triangle are approximately 36.9° and 53.1°.)

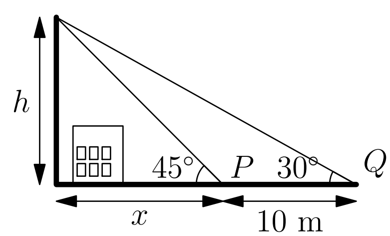
Exercise **??**: You should be able to check this for yourself, using your calculator.

Exercise **??**:

	A	B	C	a	b	c
(a)	30°	60°	90°	$2\sqrt{3}$	6	$4\sqrt{3}$
(b)	45°	45°	90°	13	13	$13\sqrt{2}$
(c)	60°	30°	90°	$5\sqrt{3}$	5	10
(d)	45°	45°	90°	$6\sqrt{2}$	$6\sqrt{2}$	12
(e)	*	*	*	*	*	*
(f)	0°	90°	90°	0	4	4

(e) is not possible: $\sqrt{2} \approx 1.41 > 1$; this would require a leg to be longer than the hypotenuse.

Exercise **??**:



Let the height of the top of the pole above eye level be h, and let the unknown distance OP be x. We can make double use of the relation $a = b\tan\theta$. In $\triangle OPT$, we have $h = OP\tan 45^\circ = x\tan 45^\circ = x$. In $\triangle OQT$, we have $h = OQ\tan 30^\circ = (x+10)\tan 30^\circ = (x+10)(\frac{1}{\sqrt{3}})$.

Multiplying the second equation throughout by $\sqrt{3}$ gives

$$h\sqrt{3} = x + 10$$

Substituting $x = h$ from the first equation gives $\sqrt{3} = h + 10$, and so

$$h = \frac{10}{\sqrt{3} - 1}$$

Putting $\sqrt{3} \approx 1.73$ gives $h \approx \frac{10}{.73} \approx 13.7$ m.

[Alternatively, we could have put $x = h\cot 45^\circ$, $x + 10 = h\cot 30^\circ$, and eliminated x by subtraction right away. Here we've used angles for which you know the values of the trig functions. But you could solve any similar problem with arbitrary angles. Suppose we put $\angle OPT = \alpha$, $\angle OQT = \beta$, $PQ = d$. Then you could put

$$h = x \tan\alpha \Leftrightarrow h = h\cot\alpha$$
$$h = (x+d)\tan\beta \Leftrightarrow (x+d) = h\cot\beta$$

Using the cotangents is more direct. You can verify that the result is $h = d/(\cot\alpha - \cot\beta)$. *Or*, you could have tackled this particular problem (with its angles of 45° and 30°) in a different way, by using the known ratios of the sides in triangles OPT and OQT. Do this for the experience! (But of course this cannot be used as a general method.)]

Exercise **??**: 2π; $\pi/2$; $3\pi/2$; $\pi/4$; $\pi/3$; 3.

Exercise **??**: (a) $\pi/6$; (b) $3\pi/4$; (c) $7\pi/6$; (d) $7\pi/4$; (e) $8\pi/9$; (f) $\pi/18$

Exercise **??**: (a) 60°; (b) 100°; (c) 7.5°; (d) 45°; (e) 210°

Exercise **??**: (a) $\sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2}$; (b) $\cos\frac{\pi}{4} = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$; (c) We won't take 'no' for an answer! — $\sin\frac{\pi}{18} = \cos(\frac{\pi}{2} - \frac{\pi}{18}) = \cos(\frac{8\pi}{18}) = \cos\frac{4\pi}{9}$.

Exercise **??**: You should be able to check these results for yourself.

Exercise **??**: You will already have checked these results.

Exercise **??**: (a) $-\cos\theta$; (b) $\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$; (c) $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$; (d) $\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$. (Note that the results of (b) and (d) are the same, because $(\theta - \frac{\pi}{4})$ and $(\theta + \frac{7\pi}{4})$ are separated by 2π.)

Exercise **??**: You should have no trouble obtaining the stated results from the preceding formulas; it's just algebra. Think about the ambiguities of sign, though.

Exercise **??**: 0.38; 0.92. (Get more significant digits if you like, and check with your calculator.)

Exercise **??**: $B = 15^\circ$, $b = 5.2$, $c = 14.1$.

Exercise **??**: $A = 56.8^\circ$, $C = 73.2^\circ$, $a = 13.1$.

Exercise **??**: $c = 14.0$, $A = 14.6^\circ$, $B = 30.3^\circ$.

Exercise **??**: $A = 22.3^\circ$, $B = 49.5^\circ$, $C = 108.2^\circ$.

Exercise **??**: $\sin C = \frac{5}{6}$, permitting $C = 56.4^\circ$, $B = 93.6^\circ$, $b = 12$, *or* $C = 123.6^\circ$, $B = 26.4^\circ$, $b = 5.3$.

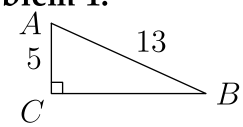
This module is based in large part on an earlier module prepared by the Department of Mathematics.

4 Trigonometry Problems and Solutions

4.1 Right triangles and trigonometric functions

(See Section **??** of the review module.)

Problem 1:



In the right triangle shown, what is $\sin A$? $\cos A$? $\tan A$? $\sec A$?

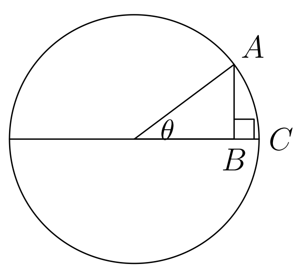
Problem 2: One of the trigonometric functions is given: find the others:

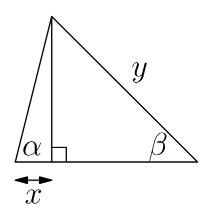
a) $\sin\theta = \frac{2}{5}$; what is $\cos\theta$? $\tan\theta$? $\sec\theta$? $(0 < \theta < 90^\circ)$

b) $\tan A = \frac{3}{4}$; what is $\sin A, \cos A, \sec A$? $(0 < A < 90^\circ)$

c) $\sec\alpha = 1.5$; what is $\sin\alpha, \cos\alpha, \tan\alpha$? $(0 < \alpha < 90^\circ)$

Problem 3: In the circle of radius 1 pictured, express the lengths of AB and BC in terms of θ.





Problem 4: In the diagram on the right, express y in terms of x, α, β.

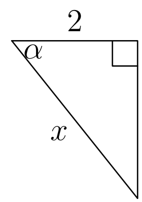
Problem 5: A wire is connected to the top of a vertical pole 7 meters high. The wire is taut, and is fastened to a stake at ground level 24 meters from the base of the pole, making an angle β with the ground. What is $\sin\beta$?

Problem 6: A ski slope rises 4 vertical feet for every 5 horizontal feet. Find $\cos\theta$, where θ is the angle of inclination.

Problem 7: In the triangle shown, express in terms of α:

a) side x

b) the area.



4.2 Special values of the trigonometric functions

(See Section **??**, of the review module.)

Problem 8: Give the values of each of the following (the angles are all in degrees):

a) $\sin 45$	e) $\tan 135$	i) $\sin 150$
b) $\tan 120$	f) $\sin(-45)$	j) $\cos 180$
c) $\tan 30$	g) $\sec 225$	k) $\sec 60$
d) $\cos 150$	h) $\cos 60$	l) $\cos(-30)$

Problem 9: An equilateral triangle has side a. Express in terms of a the distance from the center of the triangle to the midpoint of one side.

Problem 10: Without a calculator, which is bigger: $\cos 28°$ or $\cos 2.8°$?

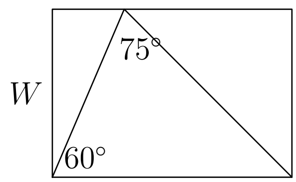
Problem 11: A regular hexagon has sides of length k. What is its height, if it is placed:

a) so a vertex is at the bottom

b) so a side is at the bottom?

Problem 12: A flashlight beam has the shape of a right circular cone with a vertex angle of 30 degrees. The flashlight is 2 meters from a vertical wall, and the beam shines horizontally on the wall so that its central axis makes a 45 degree angle with the wall. What is the horizontal width of the illuminated region on the wall? (Begin by sketching a view from above, looking down on the cone of light.)

Problem 13:



A rectangular piece of paper having width W is to be cut so that two folds can be made at 60 and 75 degree angles, as pictured. What should the length be?

Problem 14: An observer looks at a picture on a wall 10 meters away. The bottom of the picture subtends an angle 30 degrees below the horizontal; the top subtends an angle 45 degrees above the horizontal. What is the picture's vertical dimension?

4.3 Radian measure

(See Section **??** of the review module.)

Problem 15: Convert to radians the following angles in degrees: 60, 135, 210, -180, 380.

Problem 16: Convert to degrees the following angles in radians: $\pi/6$, $4\pi/9$, $3\pi/4$, $5\pi/3$.

Problem 17: Give the value of (angles are in radians): $\sin \pi/4$, $\cos 5\pi/6$, $\tan -5\pi/4$.

Problem 18: Which angle is larger: $\frac{\pi}{5}$ radians or 40 degrees?

Problem 19: The radius of a pizza slice is 25 cm and the length along the curved edge is 15cm. What is the angle of the slice in radians?

Problem 20: Fred runs around a large circular race track with a radius of 90 meters. He runs a total of 20 radians. How many complete turns around the track does he make, and how far does he go?

Problem 21: How many radians does the hour hand of a clock cover between 2:00 PM today and 5:00 AM the day after tomorrow?

Problem 22: Mimi sees a building on the horizon 6km away. To her eye, the building subtends an angle of .01 radians above the horizon. About how tall is it?

Problem 23: A wheel is spinning at 2 radians/second. How many seconds will it take to make 10 complete revolutions?

4.4 Trigonometric graphs

(See Section **??** of the review module.)

Note: radian measure is used throughout these problems.

Problem 24: Find the smallest positive solution to $\cos 3x = 0$.

Problem 25: Find the smallest positive x for which $\cos \frac{x}{2}$ has a minimum (low) point.

Problem 26: Find the smallest positive x for which $\sin 2(x + \frac{\pi}{4}) = -1$.

4.5 Trigonometric identities

(See Section **??** of the review module.)

Problem 27: If $\sin\theta = \frac{3}{5}$, what is the value of $\sin 2\theta$ and $\cos 2\theta$? ($0 < \theta < 90°$)

Problem 28: Write down the formula for $\sin(a + b)$; use it to show that $\sin(x + \frac{\pi}{2}) = \cos x$.

Problem 29: Express $\cos 2x$ in terms of $\sin x$.

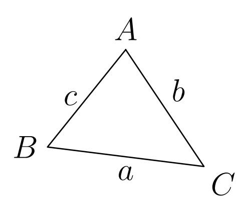
Problem 30: Simplify:

a) $\dfrac{(1 - \sin^2 x)\tan x}{\cos x}$ b) $\dfrac{\cos^2 x - 1}{\sin 2x}$

4.6 Law of Sines and Law of Cosines

(See Section **??** of the review module.)

Note: all angles are in degrees. The first five problems use the diagram below.



Problem 31: If $A = 30, B = 135$, and $b = 15$, what is a?

Problem 32: If $A = 45$, $a = 16$, and $c = 12$, what is $\sin C$?

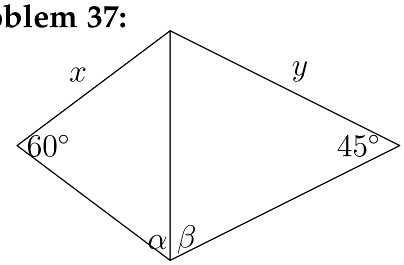
Problem 33: If $a = 5$, $b = 7$, and $c = 10$, what is $\cos C$?

Problem 34: If $a = 7$, $b = 9$, and $C = 45$, what is c?

Problem 35: If $C = 60$, express c in terms of a and b, without trig functions in your final answer.

Problem 36: Fred and John are 100 meters apart and want to measure the distance from a pole. Fred measures the angle between John and the pole to be α degrees; John measures the angle between Fred and the pole to be β degrees. Both angles are greater than 0. What is the distance between Fred and the pole?

Problem 37:



What is y in terms of x, α, and β? (Hint: use the law of sines twice.)

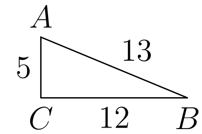
Problem 38: A surveyor is 600 meters from one end of a lake and 800 meters from the other end. From his point of view, the lake subtends an angle of 60 degrees. How long is it from one end of the lake to the other?

Problem 39: A laser on a mountaintop shines due north on a detector at sea level. There the laser beam makes an angle of 45 degrees with the ground. Then the laser shines on a second detector, also due north and at sea level, which is 4200 meters north of the first detector. At the second detector, the beam makes an angle of 15 degrees with the ground. How far is the second detector from the laser?

Problem 40: A plane is 1km from one landmark and 2km from another. From the plane's point of view, the land between them subtends an angle of 45 degrees. How far apart are the landmarks?

4.7 Solutions

Solution 1:

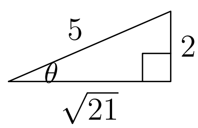


By the Pythagorean theorem: $BC = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$. Thus, $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$, $\sec A = \frac{13}{5}$.

Solution 2: There are two equivalent methods. The best is to draw a triangle and find its third side by the Pythagorean theorem, as at left. The other way is to use trigonometric identities (right).

TRIANGLE | IDENTITIES

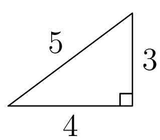
a)



Since $\sin\theta = \frac{2}{5}$, draw triangle as shown. $\sqrt{5^2 - 2^2} = \sqrt{21}$ gives the third side. Thus, $\cos\theta = \frac{\sqrt{21}}{5}$, $\tan\theta = \frac{2}{\sqrt{21}}$, $\sec\theta = \frac{5}{\sqrt{21}}$

$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \sqrt{1 - 4/25} = \sqrt{21}/5$.
Thus, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2/5}{\sqrt{21}/5} = \frac{2}{\sqrt{21}}$ and $\sec\theta = \frac{1}{\cos\theta} \Rightarrow \sec\theta = \frac{5}{\sqrt{21}}$.

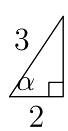
b)



Fill in 5 by the Pythagorean Theorem. $\sin A = 3/5$, $\cos A = 4/5$, $\sec A = 5/4$.

As above, but use $\sec^2 A = 1 + \tan^2 A \Rightarrow \sec A = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$. $\cos A = \frac{1}{\sec A} = \frac{4}{5}$ and $\sin A = \sqrt{1 - \cos^2 A} = \frac{3}{5}$.

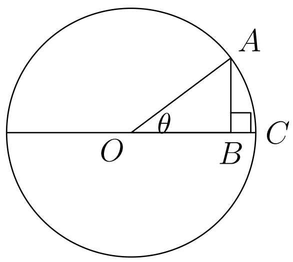
c)

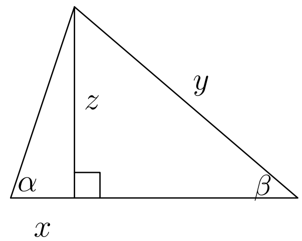


Fill in $x = \sqrt{3^2 - 2^2} = \sqrt{5}$. Thus, $\sin\alpha = \sqrt{5}/3$, $\cos\alpha = \frac{2}{3}$, $\tan\alpha = \frac{\sqrt{5}}{2}$.

Or, $\cos\alpha = \frac{1}{\sec\alpha} = \frac{2}{3}$, $\sin\alpha = \sqrt{1 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$, $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sqrt{5}/3}{2/3} = \frac{\sqrt{5}}{2}$.

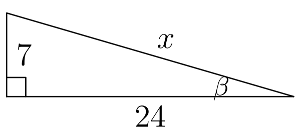
Solution 3: $\frac{AB}{1} = \sin\theta$ and $OB = \cos\theta$, so $BC = 1 - \cos\theta$.

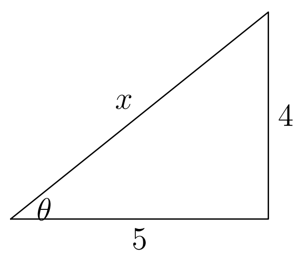




Solution 4: $\frac{z}{x} = \tan\alpha \to z = x\tan\alpha$. Also, $\frac{z}{y} = \sin\beta \to z = y\sin\beta$. Thus, $x\tan\alpha = y\sin\beta \to y = \frac{x\tan\alpha}{\sin\beta}$.

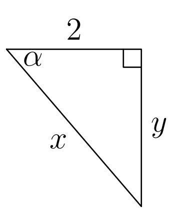
Solution 5: By the Pythagorean theorem, $x = 25$, so $\sin\beta = \frac{7}{25}$.





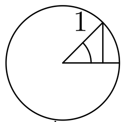
Solution 6: By the Pythagorean theorem, $x = \sqrt{4^2 + 5^2} = \sqrt{41}$ so $\cos\theta = \frac{5}{\sqrt{41}} = \frac{5\sqrt{41}}{41}$.

Solution 7:

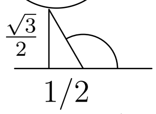


$\cos\alpha = \frac{2}{x}$, so $x = \frac{2}{\cos\alpha} = 2\sec\alpha$. To find area, $\frac{y}{2} = \tan\alpha \Rightarrow y = 2\tan\alpha$; area is $\frac{1}{2} \cdot 2y = \boxed{2\tan\alpha}$.
Other solutions: $\frac{y}{x} = \sin\alpha \Rightarrow y = 2\sec\alpha\sin\alpha = \frac{2}{\cos\alpha}\sin\alpha = 2\tan\alpha$ etc.
Or $y = \sqrt{2^2 - x^2} \Rightarrow y = \sqrt{2^2 - 2^2\sec^2\alpha} = 2\sqrt{1 - \sec^2\alpha} = 2\tan\alpha$ etc.

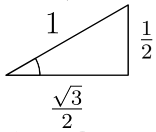
Solution 8: Best to draw the standard unit circle, put in the angle, then use your knowledge of the 30-60-90 or 45-45-90 triangle. Watch signs!

a) 

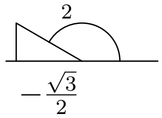
$\sin 45° = \frac{\sqrt{2}}{2}$

b) 

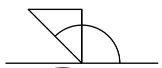
$\tan 120 = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$

c) 

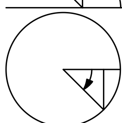
$\tan 30 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

d) 

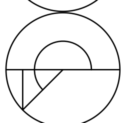
$\cos 150 = -\frac{\sqrt{3}}{2}$

e) 

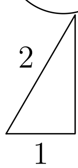
$\tan 135 = -1$

f) 

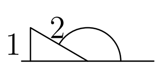
$\sin(-45) = -\frac{\sqrt{2}}{2}$ (or $\sin -45 = -\sin 45 = -\frac{\sqrt{2}}{2}$)

g) 

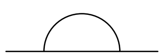
$\sec 225 = \frac{1}{-\frac{\sqrt{2}}{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

h) 

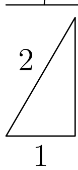
$\cos 60 = \frac{1}{2}$

i) 

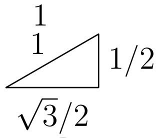
$\sin 150 = \frac{1}{2}$

j) 

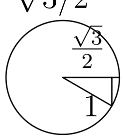
$\cos 180 = \frac{-1}{1} = -1$

k) 

$\sec 60 = \frac{2}{1} = 2$

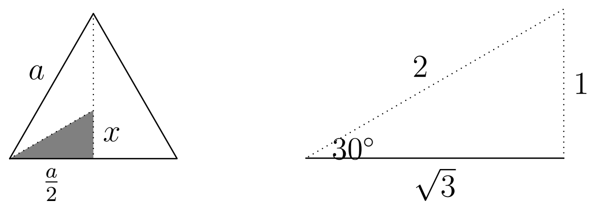
l) 

$\cos(-30) = \cos 30 = \frac{\sqrt{3}}{2}$



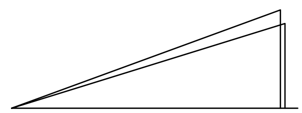
or directly.

Solution 9:

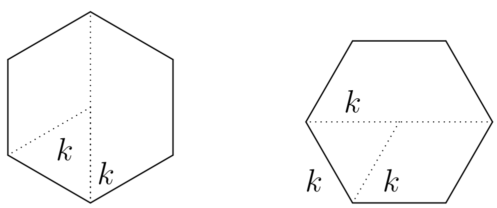


$\frac{x}{a/2} = \frac{1}{\sqrt{3}}$ since the shaded triangle is a 30-60-90 triangle; thus $x = \frac{a}{2\sqrt{3}} = \frac{a\sqrt{3}}{6}$.

Solution 10:
$\cos x$ decreases in the first quadrant as x increases. $\cos 28^\circ > \cos 30^\circ = \frac{\sqrt{3}}{2} > 0.8$ (since on squaring, $\frac{3}{4} > (0.8)^2 = .64$.)

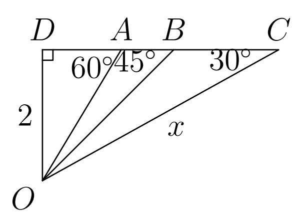


Solution 11:

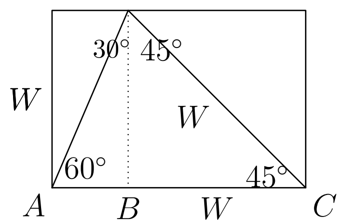


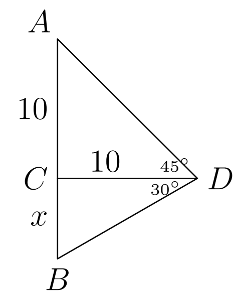
Each triangular partition is equilateral, so the height resting on a tip is $2k$. Similarly, since the height of the a 30-60-90 triangle with hypotenuse k is $k\sqrt{3}/2$, the height resting on an edge is $k\sqrt{3}$.

Solution 12:
$\angle AOB = 30^\circ \to \angle AOC = 15^\circ$. Also, $\angle DOC = 45^\circ \to \angle DOA = 30^\circ, \angle DOB = 60^\circ$.
Thus $DB = 2\sqrt{3}$
$\frac{x}{2} = \frac{1}{\sqrt{3}} \to DA = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. $AB = DB - DA = 3$.



Solution 13: The angles are as shown when the dashed perpendicular line is drawn. Using what we know about 30-60-90 and 45-45-90 triangles we deduce that $BC = w$ and $\frac{AB}{w} = \frac{1}{\sqrt{3}}$. Hence $AB = \frac{w}{\sqrt{3}} = \frac{w\sqrt{3}}{3}$, and so length is $AC = w\left(1 + \frac{\sqrt{3}}{3}\right)$.





Solution 14: $\angle ADC = 45°$, $\angle BDC = 30°$; $AB = ?$

$\frac{x}{10} = \frac{1}{\sqrt{3}} \to x = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$.

$AB = AC + BC = 10 + \frac{10}{3}\sqrt{3} = 10\left(1 + \frac{\sqrt{3}}{3}\right)$.

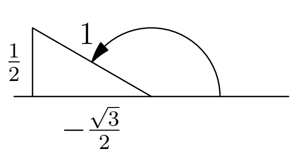
Solution 15: Multiply each number by $\frac{\pi}{180}$ to get radians.

Degrees	Radians
60	$\frac{\pi}{3}$
135	$\frac{3\pi}{4}$
210	$\frac{7\pi}{6}$
-180	$-\pi$
380	$\frac{19\pi}{9}$

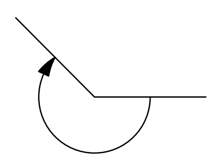
Solution 16: Multiply each number by $\frac{180}{\pi}$ to get degrees.

Radians	Degrees
$\frac{\pi}{6}$	30
$\frac{4\pi}{9}$	80
$\frac{3\pi}{4}$	135
$\frac{5\pi}{3}$	300

Solution 17:



or



$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$
$\tan\left(-\frac{5\pi}{4}\right) = \frac{1}{-1} = -1$

Solution 18: $\frac{\pi}{5}$ radians is $36°$, and so $\frac{\pi}{5}$ is a smaller angle, so $40°$ is larger.

Solution 19: $\frac{15}{25} = \frac{3}{5}$ radian (since 1 radius length of arc is 1 radian.)

(Longer way: Perimeter of circle is $2\pi \cdot 25$; so we multiply the fraction of the circle covered by the number of radians in a circle:

$$\frac{15}{2\pi \cdot 25} \cdot 2\pi$$

to get the same answer.)

Solution 20: 20 radians is $20 \cdot 90$ meters so $\boxed{1800}$ meters (since 1 radian is one radius length of arc).

Makes $\frac{20}{2\pi}$ turns around track, or approximately $\frac{10}{3.14} \approx 3+$, so 3 complete turns.

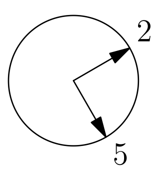
Solution 21:

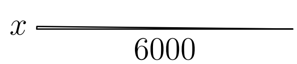
2:00pm today to 2:00pm tomorrow is 4π radians.

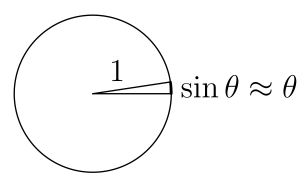
2:00pm tomorrow to 2:00am the following day is 2π radians.

2:00am that day to 5:00am that day is $\frac{\pi}{2}$ radians.

Thus, $4\pi + 2\pi + \frac{\pi}{2} = \boxed{\frac{13\pi}{2}}$ radians.





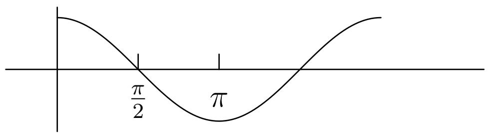


Solution 22: .01 radian is .01 radius-length of arc, so $.01 \cdot 6000$ meters is approximately 60 meters, with one significant figure (since angle is very small, arc length on circle is approximately vertical distance.)

(The exact building height is $6000 \tan .01 = 6000 \frac{\sin .01}{\cos .01} \approx 6000(.01)$ using the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ if θ is small.)

Solution 23: 10 complete revolutions is $10 \cdot 2\pi = 20\pi$ radians, so it will take $\frac{20\pi}{2} = \boxed{10\pi}$ seconds.

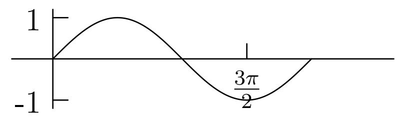
Solution 24:



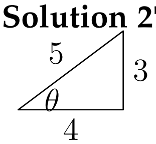
$\cos t = 0$ first when $t = \pi/2$ so $\cos 3x = 0$ first when $3x = \pi/2$, or $x = \frac{\pi}{6}$.

Solution 25: From above, $\cos t$ has its first minimum when $t = \pi$ so $\cos \frac{x}{2}$ has its first minimum when $\frac{x}{2} = \pi$, or $x = 2\pi$.

Solution 26:
$\sin t = -1$ first when $t = 3\pi/2$ so $\sin 2\left(x + \frac{\pi}{4}\right) = -1$ first when $2(x + \pi/4) = 3\pi/2$, or $x = \pi/2$.



Solution 27:



$\sin\theta = \frac{3}{5}$, so $\cos\theta = \frac{4}{5}$. $\sin 2\theta = 2\sin\theta\cos\theta = \frac{24}{25}$, and $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$.

Solution 28: $\sin(a+b) = \sin a \cos b + \cos a \sin b$

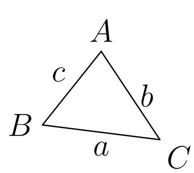
Thus, $\sin(x+\pi/2) = \sin x \cos \pi/2 + \cos x \sin \pi/2 = \sin x(0) + \cos x(1) = \cos x$.

Solution 29: $\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$

Solution 30:

$$\frac{(1-\sin^2 x)\tan x}{\cos x} = \frac{\cos^2 x}{\cos x}\frac{\sin x}{\cos x} = \sin x$$

$$\frac{\cos^2 x - 1}{\sin 2x} = \frac{-\sin^2 x}{2\sin x \cos x} = \frac{-\sin x}{2\cos x} = -\frac{\tan x}{2}$$



Solution 31: $\frac{a}{b} = \frac{\sin A}{\sin B} \rightarrow \frac{a}{15} = \frac{\sin 30}{\sin 135} = \frac{1/2}{\sqrt{2}/2} = \frac{\sqrt{2}}{2}$; $\boxed{a = \frac{15\sqrt{2}}{2}}$

Solution 32: $\frac{a}{c} = \frac{\sin A}{\sin C} \rightarrow \frac{16}{12} = \frac{\sqrt{2}/2}{\sin C}$; $\sin C = \frac{12}{16} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{3\sqrt{2}}{8}}$

Solution 33: $c^2 = a^2 + b^2 - 2ab\cos C$ by the law of cosines; thus $10^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cos C$ so $\cos C = \frac{-26}{70} = \boxed{-\frac{13}{35}}$.

Solution 34: $c^2 = a^2 + b^2 - 2ab\cos C \rightarrow c^2 = 7^2 + 9^2 - 2 \cdot 7 \cdot 9 \cdot \frac{\sqrt{2}}{2} = 130 - 63\sqrt{2}$, since $\cos 45 = \sqrt{2}/2$; thus $c = \sqrt{130 - 63\sqrt{2}}$.

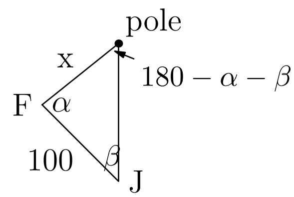
Solution 35: $c^2 = a^2 + b^2 - 2ab \cdot 12 \rightarrow c = \sqrt{a^2 + b^2 - ab}$.

Solution 36:

Since $\sin(180 - A) = \sin A$, by the law of sines

$$\frac{x}{100} = \frac{\sin\beta}{\sin(180 - \alpha - \beta)} = \frac{\sin\beta}{\sin(\alpha + \beta)}$$

$$x = \frac{100\sin\beta}{\sin(\alpha + \beta)}$$

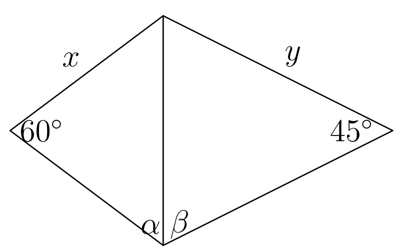


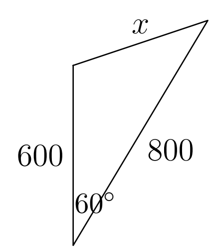
Solution 37:

$\frac{z}{x} = \frac{\sin 60}{\sin\alpha} = \frac{\sqrt{3}}{2\sin\alpha}$

$\frac{z}{y} = \frac{\sin 45}{\sin\beta} = \frac{\sqrt{2}}{2\sin\beta}$

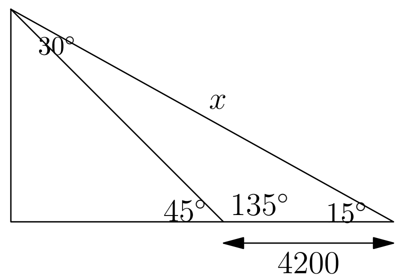
Thus, $\frac{x\sqrt{3}}{2\sin\alpha} = \frac{y\sqrt{2}}{2\sin\beta}$, or $y = \sqrt{\frac{3}{2}} \cdot \frac{\sin\beta}{\sin\alpha} \cdot x = \frac{x\sin\beta\sqrt{6}}{2\sin\alpha}$.



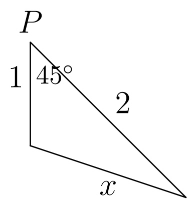


Solution 38: $c^2 = 600^2 + 800^2 - 2(600)(800)\cos 60 = 10^4(6^2 + 8^2 - 2 \cdot 6 \cdot 8 \cdot \frac{1}{2}) = 10^4(100 - 48) \rightarrow c = 100\sqrt{52} = 200\sqrt{13}$ meters.

Solution 39: By filling in other angles as shown, we get the diagram below. By the law of sines, $\frac{x}{4200} = \frac{\sin 135^\circ}{\sin 30^\circ} = \frac{\sqrt{2}/2}{1/2} = \sqrt{2} \rightarrow x = 4200\sqrt{2}$.



Solution 40:



By law of cosines, with $\cos 45^\circ = \sqrt{2}/2$,

$$x^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \sqrt{2}/2 = 5 - 2\sqrt{2}$$

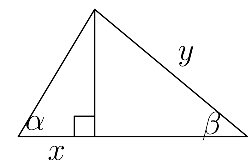
$$x = \sqrt{5 - 2\sqrt{2}}$$

5 Trigonometry Self-Tests and Solutions

5.1 Trigonometry Diagnostic Test #1

Problem 41:

Express y in terms of x, α, β.



Problem 42: An equilateral triangle has sides of length a. what is the perpendicular distance from its center to one of its sides?

Problem 43: Fred runs around a large circular race track with a radius of 900 meters. He runs a total of 20 radians. How many complete turns around the track does he make and how far does he go?

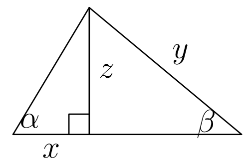
Problem 44: If $\sin\theta = \frac{2}{5}$, what is $\cos\theta$? $\sin 2\theta$?

Problem 45: A laser on top of a mountain shines due north and downward on a detector at sea level. There, the laser beam makes an angle of 45° with the ground. Then the laser shines on a second detector, also due north and at sea level, which is 4200 meters north of the first detector. At the second detector, the beam makes an angle of 15° with the ground. How far is the second detector from the laser?

5.2 Solutions to Trigonometry Diagnostic Test #1

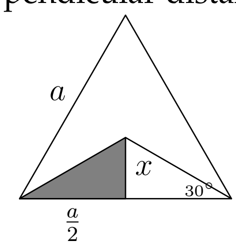
Solution 41:

Express y in terms of x, α, β.



Call the altitude z. Then $\frac{z}{x} = \tan\alpha$ and $\frac{z}{y} = \sin\beta$, so $x\tan\alpha = y\sin\beta$, so $y = \frac{\tan\alpha}{\sin\beta}x$.

Solution 42: An equilateral triangle has sides of length a. what is the perpendicular distance from its center to one of its sides?



$\frac{x}{a/2} = \tan 30° = \frac{1}{\sqrt{3}} \to x = \frac{a}{2\sqrt{3}} = \frac{a\sqrt{3}}{6}$.

Solution 43: Fred runs around a large circular race track with a radius of 900 meters. He runs a total of 20 radians. How many complete turns around the track does he make and how far does he go?

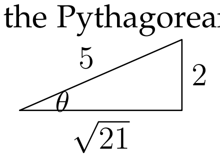
Approach 1: 1 radian = radius length on circle, so he runs $20 \cdot 900 = 18000$ meters. Me makes $\frac{20}{2\pi} = 3+$ complete turns.

Approach 2: Calculate distance by:

$$\frac{20}{2\pi} \text{ turns } \cdot (2\pi \cdot 900) \text{ meters in the circumference } = 20 \cdot 900.$$

Solution 44: If $\sin\theta = \frac{2}{5}$, what is $\cos\theta$? $\sin 2\theta$?

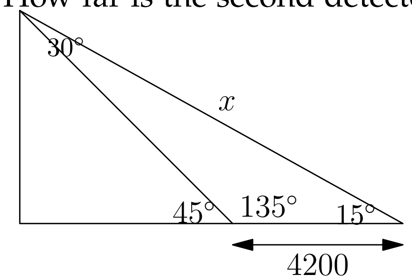
$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$, *or* draw the triangle and apply the Pythagorean theorem:



We see that $\cos\theta = \frac{\sqrt{21}}{5}$ and so
$\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{2}{5} \cdot \frac{\sqrt{21}}{5} = \frac{4\sqrt{21}}{25}$.

Solution 45: A laser on top of a mountain shines due north and downward on a detector at sea level. There, the laser beam makes an angle of

45° with the ground. Then the laser shines on a second detector, also due north and at sea level, which is 4200 meters north of the first detector. At the second detector, the beam makes an angle of 15° with the ground. How far is the second detector from the laser?



angle $a = 135^\circ$, $b = 30^\circ$.
Use law of sines:
$\frac{x}{\sin a} = \frac{4200}{\sin b}$, so $x = \frac{4200 \cdot \frac{\sqrt{2}}{2}}{\frac{1}{2}} = 4200\sqrt{2}$

5.3 Trigonometry Diagnostic Test #2

Problem 46: If $C = 90°$, $A < 90°$, and $\sin A = \frac{3}{5}$, what is $\tan A$? $\sec A$?

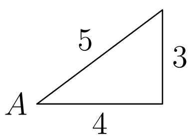
Problem 47: A flashlight shines a beam whose diameter spans an angle of $30°$ onto a wall x meters away. The axis of the flashlight makes a horizontal angle of $45°$ with the wall. What is the horizontal width of the beam on the wall?

Problem 48: Write the formula for $\sin(a + b)$ and use it to show that $\sin\left(a + \frac{\pi}{2}\right) = \cos a$.

Problem 49: Fred and John are 100 meters apart, and want to measure the distance from a pole. Fred measures the angle between John and the pole to be α degrees. John measures the angle between Fred and the pole to be β degrees. Both α and β are greater than zero. What is the distance between Fred and the pole?

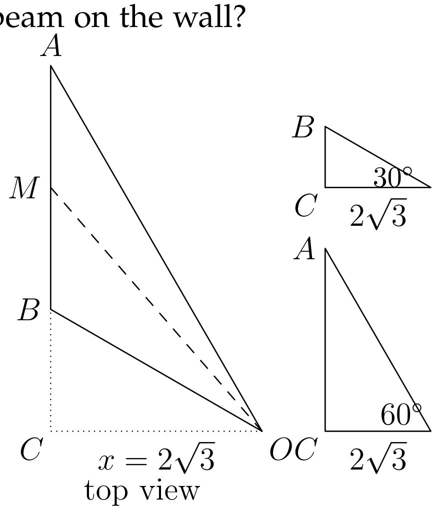
5.4 Solutions to Trigonometry Diagnostic Test #2

Solution 46: If $C = 90°$, $A < 90°$, and $\sin A = \frac{3}{5}$, what is $\tan A$? $\sec A$?



Draw triangle: $\tan A = \frac{3}{4}$, $\sec A = \frac{5}{4}$
Or, $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - (3/5)^2} = 4/5$; $\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$, $\sec A = \frac{1}{\cos A} = \frac{5}{4}$.

Solution 47: A flashlight shines a beam whose diameter spans an angle of $30°$ onto a wall x meters away. The axis of the flashlight makes a horizontal angle of $45°$ with the wall. What is the horizontal width of the beam on the wall?



$\angle AOB = 30° \to \angle MOB = 15°$
$\angle OMB = 45° \to \angle COM = 45°$

These two imply $\angle BOC = 30°$; thus $BC = 2$, and this makes $AC = (2\sqrt{3})\sqrt{3} = 6$ and thus $AB =$.

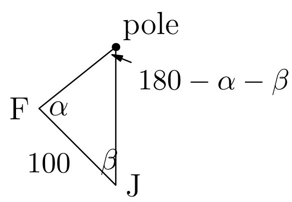
Problem 50: Write the formula for $\sin(a+b)$ and use it to show that $\sin\left(a + \frac{\pi}{2}\right) = \cos a$.

Solution 48:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a + \frac{\pi}{2}) = \sin a \cos\frac{\pi}{2} + \cos a \sin\frac{\pi}{2}$$

Since $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, this reduces to $\cos a$, as desired.

Solution 49: Fred and John are 100 meters apart, and want to measure the distance from a pole. Fred measures the angle between John and the pole to be α degrees. John measures the angle between Fred and the pole to be β degrees. Both α and β are greater than zero. What is the distance between Fred and the pole?



The third angle is $180 - \alpha - \beta$. By the law of sines:

$$\frac{x}{\sin\beta} = \frac{100}{\sin(180 - \alpha - \beta)}$$

Thus

$$x = \frac{100\sin\beta}{\sin(\alpha + \beta)}$$

since $\sin(180 - A) = \sin A$.

6 Self-Evaluation

You may want to informally evaluate your understanding of the various topic areas you have worked through in the *Self-Paced Review*. If you meet with tutors, you can show this evaluation to them and discuss whether you were accurate in your self-assessment.

For each topic which you have covered, grade yourself on a one to ten scale. One means you completely understand the topic and are able to solve all the problems without any hesitation. Ten means you could not solve any problems easily without review.

1.	Right Triangles and Trigonometric Functions	____
2.	Special Values of the Trigonometric Functions	____
3.	Radian Angle Measure	____
4.	Trigonometric Graphs	____
5.	Trigonometric Identities	____
6.	Law of sines and cosines	____