Chapter 9

Sources of Magnetic Fields

Sources of Magnetic Fields

In Chapter 8, we defined what we mean by a magnetic field and discussed at length the effect the magnetic field has on moving electric charges. In this Chapter, we consider the manner by which magnetic fields are produced.

9.1 Biot-Savart Law

Currents, which arise due to the motion of charges, are the source of magnetic fields. When charges move in a conducting wire and produce a current I, the magnetic field at any point P due to the current can be calculated by adding up the magnetic field contributions, $d\vec{\mathbf{B}}$, from small segments of the wire $d\vec{\mathbf{s}}$, (Figure 9.1.1).
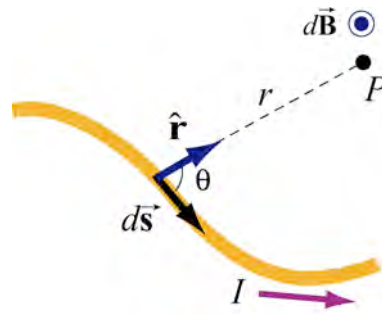
Figure 9.1.1 Magnetic field $d\vec{\mathbf{B}}$ at point P due to a current-carrying element $I\,d\vec{\mathbf{s}}$.

These segments can be thought of as a vector quantity having a magnitude of the length of the segment and pointing in the direction of the current flow. The infinitesimal current source can then be written as $I\,d\vec{\mathbf{s}}$.

Let r denote the distance from the current source to the field point P, and $\hat{\mathbf{r}}$ the corresponding unit vector. The Biot-Savart law gives an expression for the magnetic field contribution, $d\vec{\mathbf{B}}$, from the current source, $I\,d\vec{\mathbf{s}}$,

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2},\qquad (9.1.1)$$

where μ_0 is a constant called the *permeability of free space*,

$$\mu_0 = 4\pi\times10^{-7}\,\text{T}\cdot\text{m/A}. \qquad (9.1.2)$$

Notice that the expression is remarkably similar to the Coulomb's law for the electric field due to a charge element dq,

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}}. \qquad (9.1.3)$$

Adding up these contributions to find the magnetic field at the point P requires integrating over the current source,

$$\vec{\mathbf{B}} = \int_{wire} d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_{wire} \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}. \qquad (9.1.4)$$

The integral is a vector integral, which means that the expression for $\vec{\mathbf{B}}$ is really three integrals, one for each component of $\vec{\mathbf{B}}$. The vector nature of this integral appears in the cross product $I\, d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$. Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart law.

Example 9.1: Magnetic Field due to a Finite Straight Wire
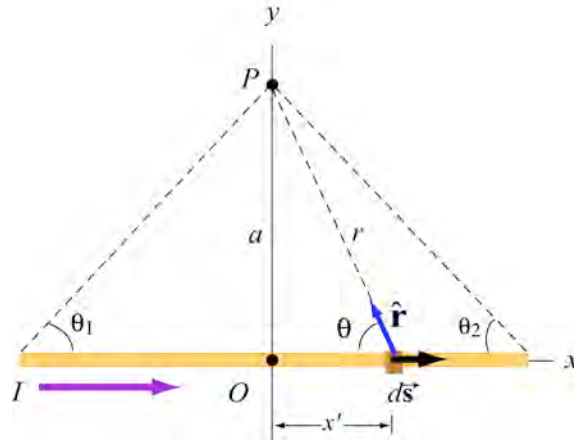
Figure 9.1.2 A thin straight wire carrying a current I.

A thin, straight wire carrying a current I is placed along the *x*-axis, as shown in Figure 9.1.2. Evaluate the magnetic field at point P. Note that we have assumed that the leads to the ends of the wire make canceling contributions to the net magnetic field at the point P.

Solution: This is a typical example involving the use of the Biot-Savart law. Consider a differential element $d\vec{\mathbf{s}} = dx'\hat{\mathbf{i}}$ carrying current I in the *x*-direction. The location of this source is represented by $\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$. Because the field point P is located at $(x, y) = (0, a)$, the position vector describing P is $\vec{\mathbf{r}}_P = a\hat{\mathbf{j}}$. The vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$ is a "relative" position vector which points from the source point to the field point. In this case, $\vec{\mathbf{r}} = a\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$, and the magnitude $r = |\vec{\mathbf{r}}| = \sqrt{a^2 + x'^2}$ is the distance from between the source and P. The corresponding unit vector is given by

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{a\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\sqrt{a^2 + x'^2}} = \sin\theta\,\hat{\mathbf{j}} - \cos\theta\,\hat{\mathbf{i}}.$$

The cross product is given by

$$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = (dx'\hat{\mathbf{i}}) \times (-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (dx'\sin\theta)\hat{\mathbf{k}}.$$

The contribution to the magnetic field due to $I\,d\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{dx\,\sin\theta}{r^2}\hat{\mathbf{k}},$$

which shows that the magnetic field at P will point in the $+\hat{\mathbf{k}}$ direction, or out of the page. The variables θ, x and r are not independent of each other. In order to complete the integration, let us rewrite the variables x and r in terms of θ. From Figure 9.1.2, we have

$$\begin{cases} r = a/\sin\theta = a\csc\theta \\ x = a\cot\theta \implies dx = -a\csc^2\theta\,d\theta. \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is

$$dB = \frac{\mu_0 I}{4\pi} \frac{(-a\csc^2\theta\,d\theta)\sin\theta}{(a\csc\theta)^2} = -\frac{\mu_0 I}{4\pi a}\sin\theta\,d\theta.$$

Integrating over all angles subtended from $-\theta_1$ to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x axis from the origin), we obtain

$$B = -\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\sin\theta\,d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1). \tag{9.1.5}$$

The first term involving θ_2 accounts for the contribution from the portion along the $+x$ axis, while the second term involving θ_1 contains the contribution from the portion along the $-x$ axis. The two terms add! Let's examine the following cases:

(i) In the symmetric case where $\theta_2 = -\theta_1$, the field point P is located along the perpendicular bisector. If the length of the rod is $2L$, then $\cos\theta_1 = L/\sqrt{L^2 + a^2}$ and the magnetic field is

$$B = \frac{\mu_0 I}{2\pi a}\cos\theta_1 = \frac{\mu_0 I}{2\pi a}\frac{L}{\sqrt{L^2 + a^2}}. \tag{9.1.6}$$

(ii) The infinite length limit $L \to \infty$. This limit is obtained by choosing $(\theta_1, \theta_2) = (0, 0)$. The magnetic field at a distance a away becomes

$$B = \frac{\mu_0 I}{2\pi a}.$$

(9.1.7)

Note that in this limit, the system possesses cylindrical symmetry, and the magnetic field lines are circular, as shown in Figure 9.1.3.
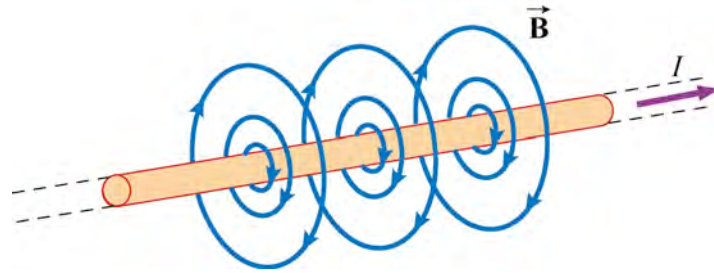
Figure 9.1.3 Magnetic field lines due to an infinite wire carrying current I.

In fact, the direction of the magnetic field due to a long straight wire can be determined by the right-hand rule (Figure 9.1.5). If you direct your right thumb along the direction of the current in the wire, then the fingers of your right hand curl in the direction of the magnetic field. In cylindrical coordinates (r, ϕ, z) where the unit vectors are related by $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$, if the current flows in the $+z$-direction, then, using the Biot-Savart law, the magnetic field must point in the φ-direction.
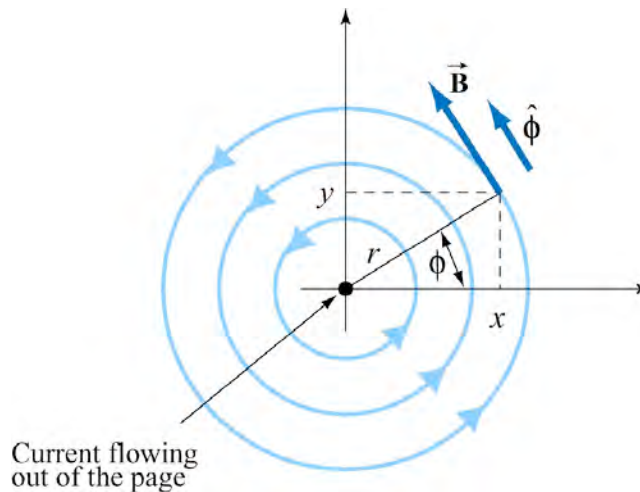
Figure 9.1.4 Direction of the magnetic field due to an infinite straight wire

Example 9.2: Magnetic Field due to a Circular Current Loop

A circular loop of radius R in the xy-plane carries a steady current I, as shown in Figure 9.1.5.

(a) What is the magnetic field at a point P on the axis of the loop, at a distance z from the center?

(b) If we place a magnetic dipole $\vec{\mu} = \mu_z \hat{k}$ at P, find the magnetic force experienced by the dipole. Is the force attractive or repulsive? What happens if the direction of the dipole is reversed, i.e., $\vec{\mu} = -\mu_z \hat{k}$?
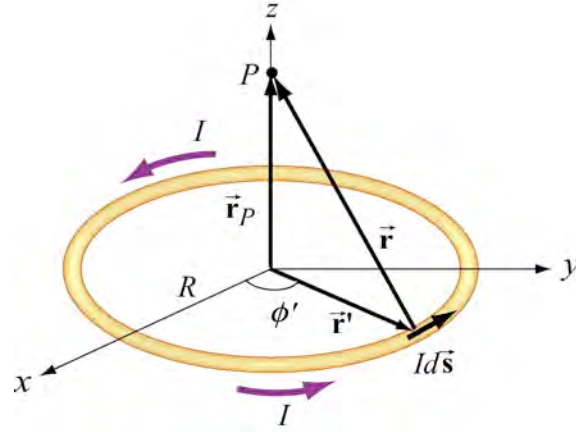
Figure 9.1.5 Magnetic field due to a circular loop carrying a steady current.

Solution:

(a) This is another example that involves the application of the Biot-Savart law. In Cartesian coordinates, the differential current element located at $\vec{r}' = R(\cos\phi'\hat{i} + \sin\phi'\hat{j})$ can be written as $Id\vec{s} = I(d\vec{r}'/d\phi')d\phi' = IRd\phi'(-\sin\phi'\hat{i} + \cos\phi'\hat{j})$. Because the field point P is on the axis of the loop at a distance z from the center, its position vector is given by $\vec{r}_P = z\hat{k}$. The relative position vector is given by

$$\vec{r} = \vec{r}_P - \vec{r}' = -R\cos\phi'\hat{i} - R\sin\phi'\hat{j} + z\hat{k}, \qquad (9.1.8)$$

Its magnitude the distance between the differential current element and P,

$$r = |\vec{r}| = \sqrt{(-R\cos\phi')^2 + (-R\sin\phi')^2 + z^2} = \sqrt{R^2 + z^2}. \qquad (9.1.9)$$

Thus, the corresponding unit vector from $Id\vec{s}$ to P can be written as

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|} .$$

The cross product $d\vec{\mathbf{s}} \times (\vec{\mathbf{r}}_P - \vec{\mathbf{r}}')$ can be simplified as

$$
\begin{aligned}
d\vec{\mathbf{s}} \times (\vec{\mathbf{r}}_P - \vec{\mathbf{r}}') &= R\,d\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \times (-R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \\
&= R\,d\phi'(z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}}).
\end{aligned}
\tag{9.1.10}
$$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$
\begin{aligned}
d\vec{\mathbf{B}} &= \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \vec{\mathbf{r}}}{r^3} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times (\vec{\mathbf{r}}_P - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|^3} \\
&= \frac{\mu_0 IR}{4\pi} \frac{z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}}}{(R^2 + z^2)^{3/2}} d\phi'.
\end{aligned}
\tag{9.1.11}
$$

Using the result obtained above, the magnetic field at P is

$$\vec{\mathbf{B}} = \frac{\mu_0 IR}{4\pi} \int_0^{2\pi} \frac{z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}}}{(R^2 + z^2)^{3/2}} d\phi'. \tag{9.1.12}$$

The x and the y components of $\vec{\mathbf{B}}$ can be readily shown to be zero,

$$B_x = \frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \int_0^{2\pi} \cos\phi'\,d\phi' = \frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \sin\phi' \Big|_0^{2\pi} = 0, \tag{9.1.13}$$

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \int_0^{2\pi} \sin\phi'\,d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \cos\phi' \Big|_0^{2\pi} = 0. \tag{9.1.14}$$

On the other hand, the z component is

$$B_z = \frac{\mu_0}{4\pi} \frac{IR^2}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi} \frac{2\pi IR^2}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2 + z^2)^{3/2}}. \tag{9.1.15}$$

Thus, we see that along the symmetric axis, B_z is the only non-vanishing component of the magnetic field. The conclusion can also be reached by using symmetry arguments. The plot of B_z / B_0 where $B_0 = \mu_0 I / 2R$ is the magnetic field strength at $z = 0$, as a function of z / R is shown in Figure 9.1.6.
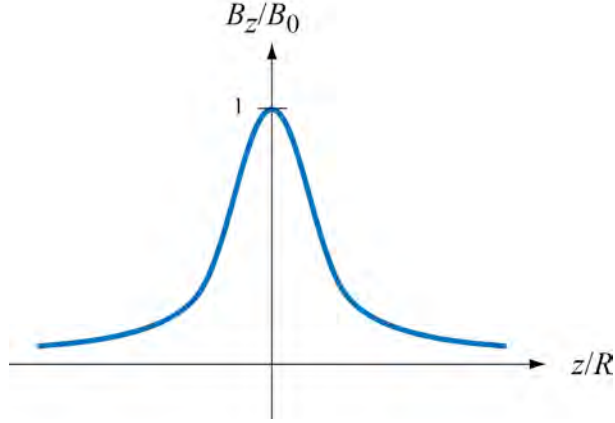
Figure 9.1.6 Plot of the ratio of the magnetic field, B_z / B_0, as a function of z / R

(b) If we place a magnetic dipole $\vec{\mu} = \mu_z \hat{\mathbf{k}}$ at the point P, as discussed in Chapter 8, due to the non-uniformity of the magnetic field, the dipole will experience a force given by

$$\vec{\mathbf{F}}_B = \nabla(\vec{\mu} \cdot \vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z \frac{dB_z}{dz} \hat{\mathbf{k}} . \tag{9.1.16}$$

Upon differentiating Eq. (9.1.15) and substituting into Eq. (9.1.16), we obtain

$$\vec{\mathbf{F}}_B = -\frac{3\mu_z \mu_0 I R^2 z}{2(R^2 + z^2)^{5/2}} \hat{\mathbf{k}} . \tag{9.1.17}$$

Thus, the dipole is attracted toward the current-carrying ring. If the direction of the dipole is reversed, $\vec{\mu} = -\mu_z \hat{\mathbf{k}}$, the resulting force will be repulsive.

9.1.1 Magnetic Field of a Moving Point Charge

Suppose we have an infinitesimal current element in the form of a cylinder of cross-sectional area A and length ds consisting of n charge carriers per unit volume, all moving at a common velocity $\vec{\mathbf{v}}$ along the axis of the cylinder. Let I be the current in the element, which we define as the amount of charge passing through any cross-section of the cylinder per unit time. From Chapter 6, we see that the current I can be written as

$$n A q |\vec{\mathbf{v}}| = I . \tag{9.1.18}$$

The total number of charge carriers in the current element is simply $dN = n A ds$. Using Eq. (9.1.1), the magnetic field $d\vec{\mathbf{B}}$ due to the dN charge carriers is given by

$$ d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \frac{(nAq\,|\,\vec{\mathbf{v}}\,|)\,d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{(n\,A\,ds)q\,\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{(dN)q\,\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}, \quad (9.1.19) $$

where r is the distance between the charge and the field point P at which the field is being measured, the unit vector $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$ points *from* the source of the field (the charge) *to* P. The differential length vector $d\,\vec{\mathbf{s}}$ is defined to be parallel to $\vec{\mathbf{v}}$. In case of a single charge, $dN = 1$, the above equation becomes

$$ \vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \frac{q\,\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}. \quad (9.1.20) $$

Because a point charge does not constitute a steady current, the above equation only holds in the non-relativistic limit, where $v \ll c$, (c is the speed of light), so that the effect of "retardation" can be ignored.

The result may be readily extended to a collection of N point charges, each moving with a different velocity. Let the ith charge q_i be located at (x_i, y_i, z_i) and moving with velocity $\vec{\mathbf{v}}_i$. Using the superposition principle, the magnetic field at P can be obtained as:

$$ \vec{\mathbf{B}} = \sum_{i=1}^{N} \frac{\mu_0}{4\pi} q_i \vec{\mathbf{v}}_i \times \left[\frac{(x - x_i)\hat{\mathbf{i}} + (y - y_i)\hat{\mathbf{j}} + (z - z_i)\hat{\mathbf{k}}}{\left[(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 \right]^{3/2}} \right]. \quad (9.1.21) $$

9.1.2 Magnetic Field of a Moving Charge Movie

Figure 9.1.7 shows one frame of the animations of the magnetic field of a moving positive and negative point charge, assuming the speed of the charge is small compared to the speed of light.

(a) http://youtu.be/JmqX1GrMYnU   (b) http://youtu.be/Apde4g619RA
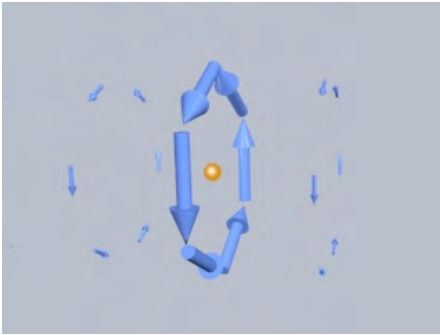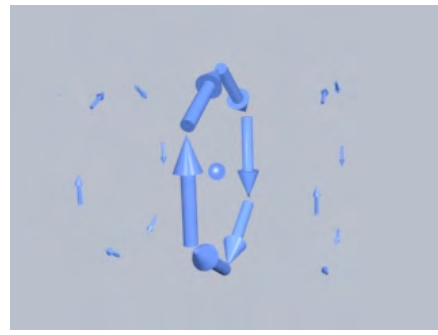
Figure 9.1.7 The magnetic field of (a) a moving positive charge and of (b) a moving negative charge when the speed of the charge is small compared to c.

9.1.3 Magnetic Field of Four Charges Moving in a Circle Movie

Suppose we want to calculate the magnetic fields of a number of charges moving on the circumference of a circle with equal spacing between the charges. To calculate this field we have to add up vectorially the magnetic fields of each of charges using Eq. (9.1.19).

http://youtu.be/-ylqucmTd00
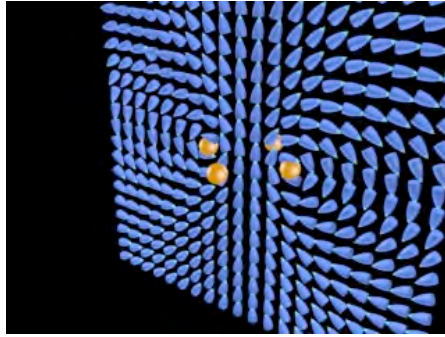
Figure 9.1.8 The magnetic field of four charges moving in a circle. We show the magnetic field vector directions in only one plane. The bullet-like icons indicate the direction of the magnetic field at that point in the array spanning the plane.

Figure 9.1.8 shows one frame of the animation when the number of moving charges is four. When the number of charges becomes eight, a characteristic pattern emerges--the magnetic dipole pattern. Far from the ring, the shape of the field lines is the same as the shape of the field lines for an electric dipole.

9.2 Force Between Two Parallel Wires

9.2.1 Forces between Current-Carrying Parallel Wires—The Experiment

(a) http://youtu.be/rU7QOukFjTo   (b) http://youtu.be/rU7QOukFjTo
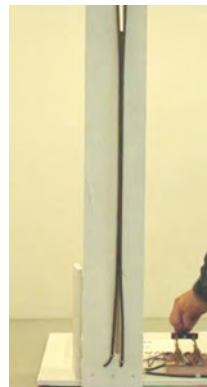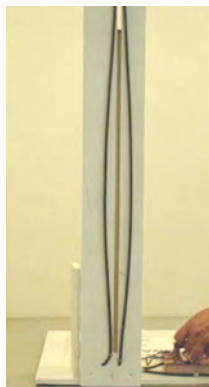
Figure 9.2.1 Current-carrying parallel wires can repel or attract each other.

Two long parallel wires carrying current in opposite directions will repel one another. This is shown in Figure 9.1.1(a). If the current is in the same direction in both wires, the wires will attract one another, as shown in Figure 9.1.1(b). The movie linked to Figure 9.1.1 shows this behavior for the two directions of current.

9.2.2 Forces between Parallel Wires—The Standard Argument

To explain this phenomenon, we first give the standard argument that you will find in almost all introductory textbooks. We have already seen that a current-carrying wire produces a magnetic field. In addition, when placed in a magnetic field, a wire carrying a current will experience a net force. Thus, we expect two current-carrying wires to exert forces on each other.

Consider two parallel wires separated by a distance a and carrying currents I_1 and I_2 in the +x-direction, as shown in Figure 9.2.1.

Figure 9.2.1 Force between two parallel wires

The magnetic force, $\vec{F}_{21}$, exerted by wire 2 on wire 1 may be computed as follows. Using the result from the previous example, the magnetic field lines due to I_2 going in the +x-direction are circles concentric with wire 2, with the field $\vec{B}_2$ pointing in the tangential directi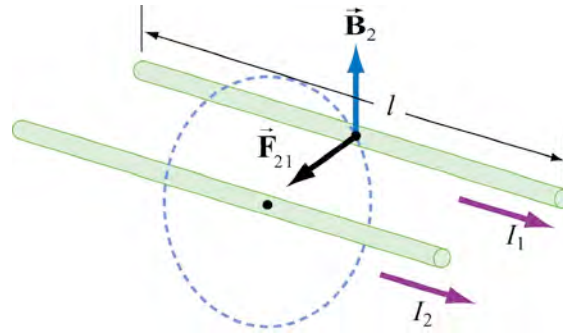on. Thus, at an arbitrary point P on wire 1, we have $\vec{B}_2 = -(\mu_0 I_2 / 2\pi a)\hat{\mathbf{j}}$, which points in the direction perpendicular to wire 1, as depicted in Figure 9.2.1. Therefore,

$$\vec{F}_{21} = I_1 \vec{l} \times \vec{B}_2 = I_1(l\hat{\mathbf{i}}) \times \left(-\frac{\mu_0 I_2}{2\pi a}\hat{\mathbf{j}} \right) = -\frac{\mu_0 I_1 I_2 l}{2\pi a}\hat{\mathbf{k}}. \qquad (9.2.1)$$

The force $\vec{F}_{21}$ points toward wire 2. The conclusion we can draw from this simple calculation is that two parallel wires carrying currents in the same direction will attract each other. On the other hand, if the currents flow in opposite directions, the resultant force will be repulsive. Note in making this standard calculation, we never talked about the total magnetic field due to both current-carrying wires. We did not need to do this to

calculate the force because the force on wire 1 due to its own magnetic field is zero, so we can ignore its presence and still get the correct answer for the force on wire 1.

9.2.3 Forces between Parallel Wires Movie—Faraday's Argument

To get a more intuitive feel for what causes this attraction or repulsion, we look at the total magnetic field of the two wires, and interpret the result we obtained above in Eq. (9.2.1) in terms of pressures and tensions transmitted between the wires by the magnetic field, as indicated by the shape of the total field lines. Figures 9.2.2 shows parallel wires carrying current in the same and in opposite directions, and the corresponding total magnetic fields. In the first case, the magnetic field configuration produces an attraction between the wires. This is because the magnetic tension between the wires pulls them together, whereas the higher magnetic pressure outside the wires pushes them together. In the second case the magnetic field configuration produces repulsion between the wires. This is because the higher magnetic pressure between the wires for pushes them apart.

(a) http://youtu.be/nQX-BM3GCv4   **(b)** http://youtu.be/5nKQjKgS9z0
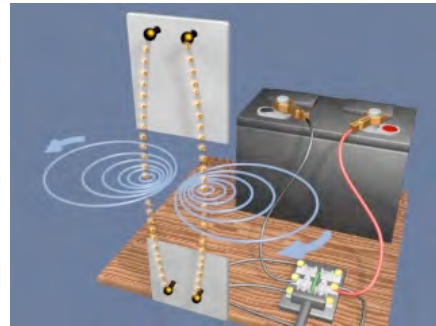
Figure 9.2.2 (a) The attraction between two wires carrying current in the same direction. The direction of current flow is represented by the motion of the orange spheres in the visualization. (b) The repulsion of two wires carrying current in opposite directions.

9.3 Ampere's Law (see also Ampere's Law Simulation in Section 9.14)

We now introduce Ampere's Law. Many of the conceptual problems students have with Ampere's Law have to do with understanding the geometry, and we urge you to read the standard development below and then go to the Ampere's Law simulation in Section 9.14. There you can interact directly with the relevant geometry in a 3D interactive simulation of Ampere's Law.

We have seen that moving charges or currents are the source of magnetism. This can be readily demonstrated by placing compass needles near a wire. As shown in Figure 9.3.1a, all compass needles point in the same direction in the absence of current. However, when $I \neq 0$, the needles will be deflected along the tangential direction of the circular path (Figure 9.3.1b).
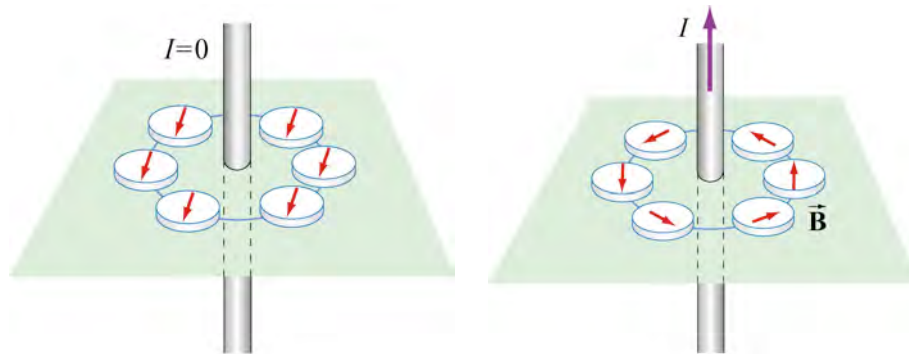
Figure 9.3.1 Deflection of compass needles near a current-carrying wire.

Let us now divide a circular path of radius r into a large number of small length vectors $\Delta \vec{s} = \Delta s \, \hat{\boldsymbol{\theta}}$, that point along the tangential direction with magnitude Δs (Figure 9.3.2).
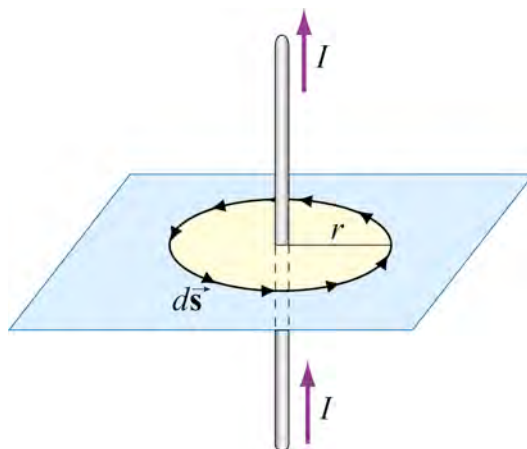
Figure 9.3.2 Amperian loop

In the limit $\Delta \vec{s} \to \vec{0}$, we obtain

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = \left(\frac{\mu_0 I}{2\pi r} \right) (2\pi r) = \mu_0 I . \tag{9.3.1}$$

The result above is obtained by choosing a closed path, called an *Amperian loop* that follows one particular magnetic field line. Let's consider a slightly more complicated Amperian loop, shown in Figure 9.3.3
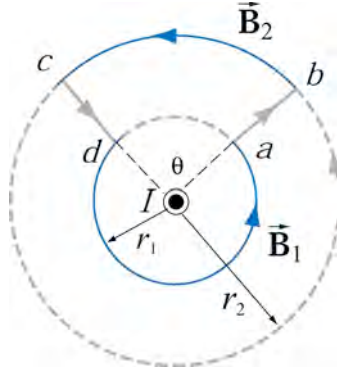
Figure 9.3.3 An Amperian loop involving two field lines.

The line integral of the magnetic field around the contour *abcda* is

$$\oint_{abcda} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \int_{ab} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_{bc} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_{cd} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_{cd} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$$

$$= 0 + B_2(r_2\theta) + 0 + B_1 r_1(2\pi - \theta),$$

(9.3.2)

where the lengths of arcs *bc* is $r_2\theta$, and *ad* is $r_1(2\pi - \theta)$. The first and the third integrals vanish since the magnetic field is perpendicular to the paths of integration. With $B_1 = \mu_0 I / 2\pi r_1$ and $B_2 = \mu_0 I / 2\pi r_2$, the above expression becomes

$$\oint_{abcda} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1} r_1(2\pi - \theta) = \frac{\mu_0 I}{2\pi}\theta + \frac{\mu_0 I}{2\pi}(2\pi - \theta) = \mu_0 I. \quad (9.3.3)$$

We see that the same result is obtained whether the closed path involves one or two magnetic field lines.

As shown in Example 9.1, in cylindrical coordinates (r, ϕ, z) with current flowing in the $+z$-axis, the magnetic field is given by $\vec{\mathbf{B}} = (\mu_0 I / 2\pi r)\hat{\boldsymbol{\phi}}$. An arbitrary length element in the cylindrical coordinates can be written as

$$d\vec{\mathbf{s}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}, \quad (9.3.4)$$

which implies

$$\oint_{\text{closed path}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \oint_{\text{closed path}} \left(\frac{\mu_0 I}{2\pi r}\right) r\,d\phi = \frac{\mu_0 I}{2\pi} \oint_{\text{closed path}} d\phi = \frac{\mu_0 I}{2\pi}(2\pi) = \mu_0 I. \quad (9.3.5)$$

In other words, the line integral of $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed Amperian loop is proportional to I_{enc}, the current encircled by the loop.
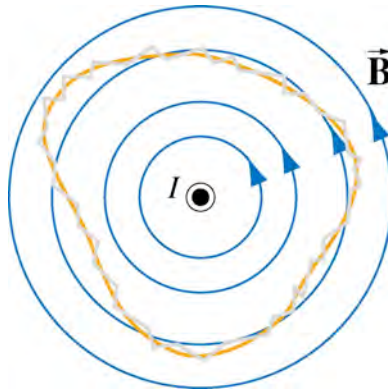
Figure 9.3.4 An Amperian loop of arbitrary shape.

The generalization to any closed loop of arbitrary shape (see for example, Figure 9.3.4) that involves many magnetic field lines is known as Ampere's law,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}.$$ (9.3.6)

Ampere's law in magnetism is analogous to Gauss's law in electrostatics. In order to apply them, the system must possess certain symmetry. In the case of an infinite wire, the system possesses cylindrical symmetry and Ampere's law can be readily applied. However, when the length of the wire is finite, Biot-Savart law must be used instead.

Biot-Savart Law	$\vec{\mathbf{B}} = \dfrac{\mu_0 I}{4\pi} \displaystyle\int \dfrac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$	general current source ex: finite wire
Ampere's law	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}$	current source has certain symmetry ex: infinite wire (cylindrical)

Ampere's law is applicable to the following current configurations:

1. Infinitely long straight wires carrying a steady current I (Example 9.3).

2. Infinitely large sheet of thickness b with a current density $\vec{\mathbf{J}}$ (Example 9.4).

3. Infinite solenoid (Section 9.4).

4. Toroid (Example 9.5).

We shall examine all four configurations in detail.

Example 9.3: Field Inside and Outside a Current-Carrying Wire

Consider a long straight wire of radius R carrying a current I of uniform current density, as shown in Figure 9.3.5. Find the magnetic field everywhere.
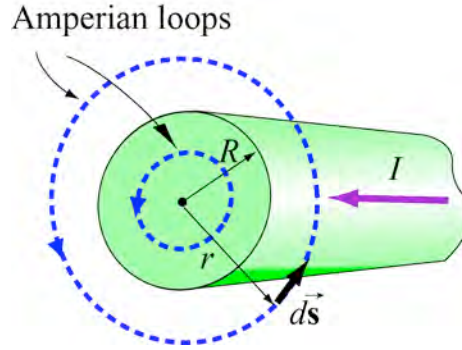
Figure 9.3.5 Amperian loops for calculating the $\vec{B}$ field of a conducting wire of radius R.

Solution:

(i) Outside the wire where $r \geq R$, the Amperian loop (circle 1) completely encircles the current, i.e., $I_{enc} = I$. Applying Ampere's law yields

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r) = \mu_0 I .$$

The magnetic field is then

$$B = \frac{\mu_0 I}{2\pi r}, \qquad r \geq R .$$

(ii) Inside the wire where $r < R$, the amount of current encircled by the Amperian loop (circle 2) is proportional to the area enclosed,

$$I_{enc} = \left(\frac{\pi r^2}{\pi R^2} \right) I .$$

Thus, we have

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_0 I \left(\frac{\pi r^2}{\pi R^2} \right) .$$

The magnetic field is then

$$B = \frac{\mu_0 I r}{2\pi R^2} , \qquad r < R .$$

We see that the magnetic field is zero at the center of the wire and increases linearly with r until $r = R$. Outside the wire, the field falls off as $1/r$. The qualitative behavior of the field is depicted in Figure 9.3.6.
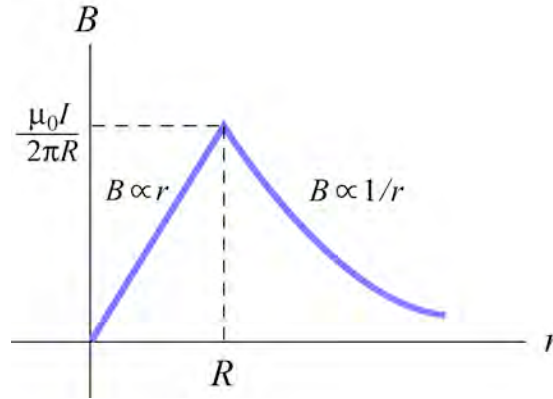
Figure 9.3.6 Magnetic field of a conducting wire of radius R carrying a steady current I.

Example 9.4: Magnetic Field Due to an Infinite Current Sheet

Consider an infinitely large sheet of thickness b lying in the xy-plane with a uniform current density $\vec{\mathbf{J}} = J_0 \hat{\mathbf{i}}$. Find the magnetic field everywhere.
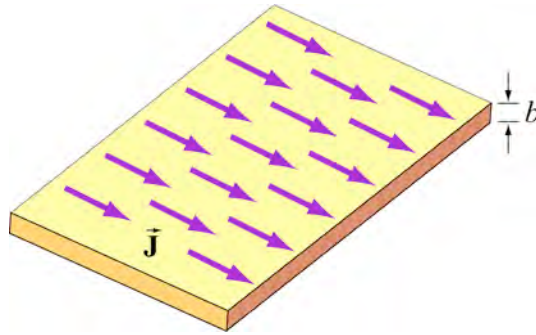
Figure 9.3.7 An infinite sheet with current density $\vec{\mathbf{J}} = J_0 \hat{\mathbf{i}}$.

Solution: We may think of the current sheet as a set of parallel wires carrying currents in the $+x$-direction. From Figure 9.3.8, we see that magnetic field at a point P above the plane points in the negative y-direction. The z-component vanishes after adding up the contributions from all wires. Similarly, we may show that the magnetic field at a point below the plane points in the positive y-direction.
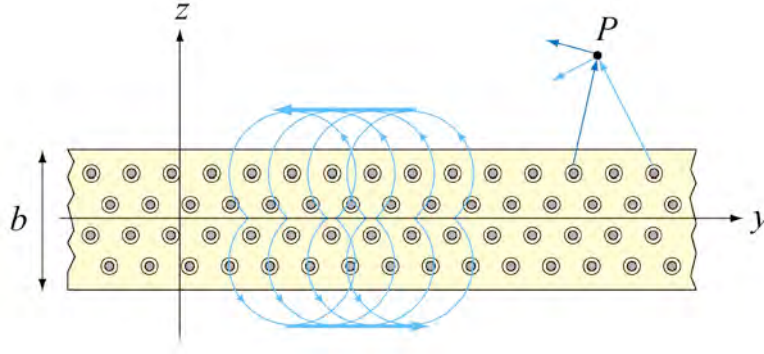
Figure 9.3.8 Magnetic field of a current sheet.

We may now apply Ampere's law to find the magnetic field due to the current sheet. The Amperian loops are shown in Figure 9.3.9.
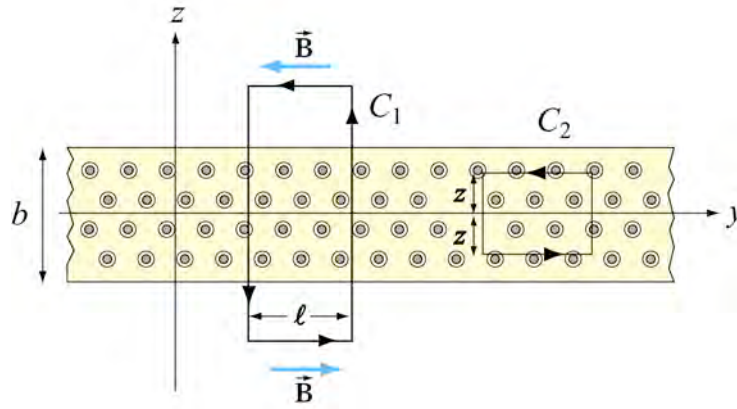
Figure 9.3.9 Amperian loops for the current sheets.

For the field outside, $|z| > b/2$, we integrate along path C_1. The amount of current enclosed by C_1 is

$$I_{enc} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(b\ell).$$

(9.3.7)

Applying Ampere's law leads to

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{enc} = \mu_0(J_0 b\ell).$$

(9.3.8)

Therefore the magnetic field is

$$B = \mu_0 J_0 b/2, \quad |z| > b/2.$$

(9.3.9)

Note that the magnetic field outside the sheet is constant, independent of the distance from the sheet. Next we find the magnetic field inside the sheet, $|z| < b/2$. The amount of current enclosed by path C_2 is

$$I_{enc} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(2\ell \,|z|) \tag{9.3.10}$$

Applying Ampere's law, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{enc} = \mu_0 J_0(2\ell \,|z|). \tag{9.3.11}$$

Therefore the magnetic field is

$$B = \mu_0 J_0 \,|z|, \qquad |z| < b/2. \tag{9.3.12}$$

At $z = 0$, the magnetic field vanishes, as required by symmetry. The results can be summarized using the unit-vector notation as

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 J_0 b}{2}\,\hat{\mathbf{j}}, & z > b/2 \\[2mm] -\mu_0 J_0 z\,\hat{\mathbf{j}}, & -b/2 < z < b/2 \\[2mm] \dfrac{\mu_0 J_0 b}{2}\,\hat{\mathbf{j}}, & z < -b/2. \end{cases} \tag{9.3.13}$$

Let's now consider the limit where the sheet is infinitesimally thin, $b \to 0$. In this case, instead of current density $\vec{\mathbf{J}} = J_0 \hat{\mathbf{i}}$, we have surface current $\vec{\mathbf{K}} = K\hat{\mathbf{i}}$, where $K = J_0 b$. Note that the dimension of K is current/length. In this limit, the magnetic field becomes

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 K}{2}\,\hat{\mathbf{j}}, & z > 0 \\[2mm] \dfrac{\mu_0 K}{2}\,\hat{\mathbf{j}}, & z < 0. \end{cases} \tag{9.3.14}$$

9.4 Solenoid

A solenoid is a long coil of wire tightly wound in the helical form. Figure 9.4.1 shows the magnetic field lines of a solenoid carrying a steady current *I*. We see that if the turns are closely spaced, the resulting magnetic field inside the solenoid becomes fairly uniform, provided that the length of the solenoid is much greater than its diameter. For an "ideal" solenoid, which is infinitely long with turns tightly packed, the magnetic field inside the solenoid is uniform and parallel to the axis, and vanishes outside the solenoid.
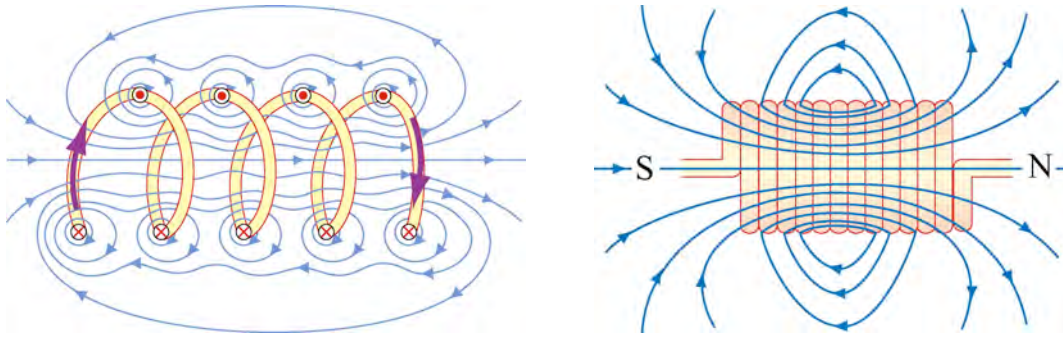
Figure 9.4.1 Magnetic field lines of a solenoid

We can use Ampere's law to calculate the magnetic field strength inside an ideal solenoid. The cross-sectional view of an ideal solenoid is shown in Figure 9.4.2.
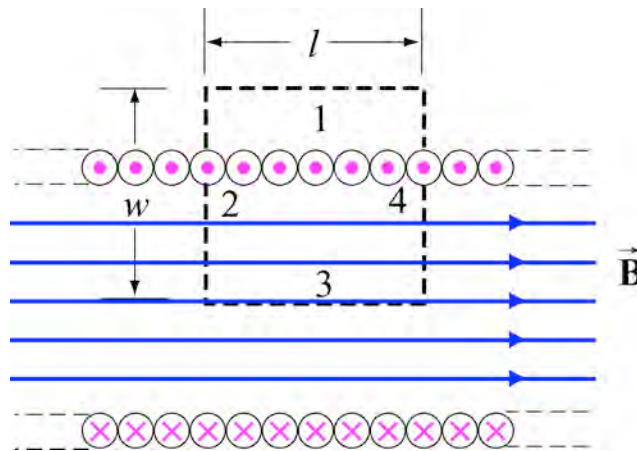
Figure 9.4.2 Amperian loop for calculating the magnetic field of an ideal solenoid.

To compute $\vec{\mathbf{B}}$, we consider a rectangular path of length l and width w and traverse the path in a counterclockwise manner. The line integral of $\vec{\mathbf{B}}$ along this loop is

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \int_1 \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_2 \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_3 \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_4 \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$$

$$= \quad 0 \quad + \quad 0 \quad + \quad Bl \quad + \quad 0.$$

(9.4.1)

In the above, the contributions along sides 2 and 4 are zero because $\vec{\mathbf{B}}$ is perpendicular to $d\vec{\mathbf{s}}$. In addition, $\vec{\mathbf{B}} = \vec{\mathbf{0}}$ along side 1 because the magnetic field is non-zero only inside the solenoid. On the other hand, the current enclosed by the Amperian loop is $I_{enc} = NI$, where N is the number of enclosed turns. Applying Ampere's law yields

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl = \mu_0 NI .$$ (9.4.2)

The magnetic field inside the solenoid is then

$$B = \frac{\mu_0 NI}{l} = \mu_0 nI ,$$ (9.4.3)

where $n = N/l$ represents the number of turns per unit length., In terms of the surface current, $K = nI$, which has dimensions of current per unit length, the magnetic field can also be written as

$$B = \mu_0 K .$$ (9.4.4)

What happens if the length of the solenoid is finite? To find the magnetic field due to a finite solenoid, we shall approximate the solenoid as consisting of a large number of circular loops stacking together.
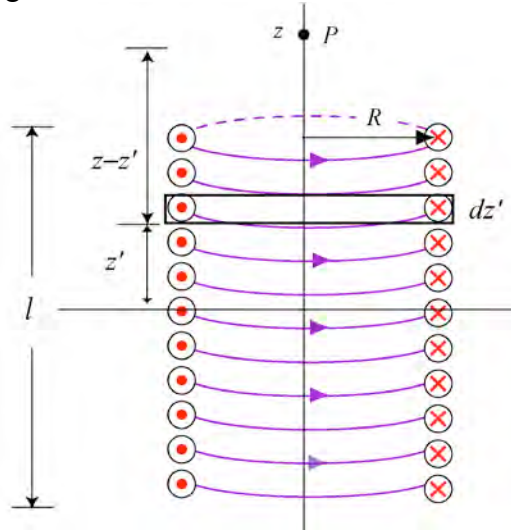
Figure 9.4.3 Finite Solenoid

Using the result obtained in Example 9.2, the magnetic field at a point P on the z axis may be calculated as follows: Take a cross section of tightly packed loops located at z' with a thickness dz', as shown in Figure 9.4.3. The amount of current is proportional to the thickness of the cross section and is given by $dI = I(ndz') = I(N/l)dz'$, where $n = N/l$ is the number of turns per unit length.

The contribution to the magnetic field at P due to this subset of loops is

$$dB_z = \frac{\mu_0 R^2}{2[(z - z')^2 + R^2]^{3/2}} dI = \frac{\mu_0 R^2}{2[(z - z')^2 + R^2]^{3/2}} (nI dz') .$$ (9.4.5)

Integrating over the entire length of the solenoid, we obtain

$$B_z = \frac{\mu_0 n I R^2}{2} \int_{-l/2}^{l/2} \frac{dz'}{[(z-z')^2 + R^2]^{3/2}} = \frac{\mu_0 n I R^2}{2} \frac{z'-z}{R^2\sqrt{(z-z')^2 + R^2}} \Bigg|_{-l/2}^{l/2}$$

$$= \frac{\mu_0 n I}{2} \left[\frac{(l/2)-z}{\sqrt{(z-l/2)^2 + R^2}} + \frac{(l/2)+z}{\sqrt{(z+l/2)^2 + R^2}} \right].$$

(9.4.6)

A plot of B_z / B_0, where $B_0 = \mu_0 n I$ is the magnetic field of an infinite solenoid, as a function of z / R is shown in Figure 9.4.4 for $l = 10R$ and $l = 20R$.
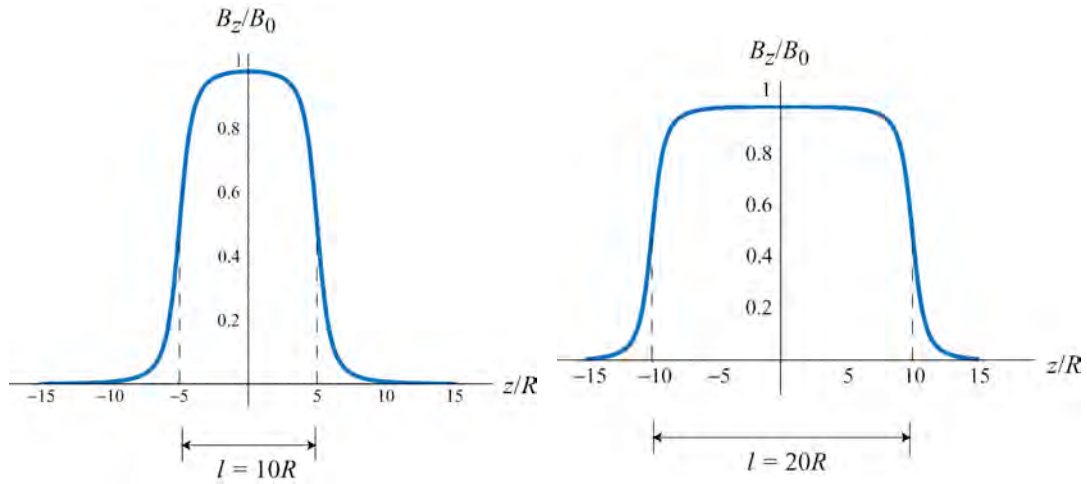
Figure 9.4.4 Plot of magnetic field of a finite solenoid for (a) $l = 10R$, and (b) $l = 20R$.

Notice that the value of the magnetic field in the region $|z| < l/2$ is nearly uniform and approximately equal to B_0.

Examaple 9.5: Toroid

Consider a toroid that consists of N turns, with inner radius a and outer radius b, as shown in Figure 9.4.5. Find the magnetic field everywhere.
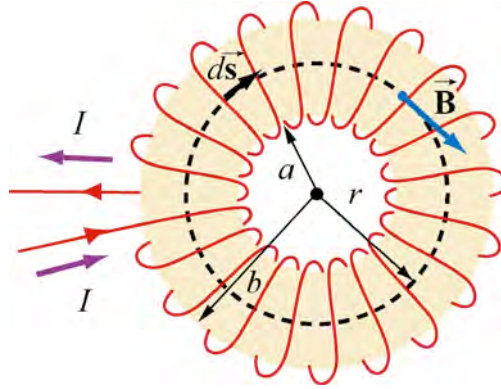
Figure 9.4.5 A toroid with N turns

Solution: One can think of a toroid as a solenoid wrapped around with its ends connected. Thus, the magnetic field is completely confined inside the toroid and the field points in the azimuthal direction (clockwise due to the way the current flows, as shown in Figure 9.4.5.)

Applying Ampere's law using a circular loop of radius r, for the region $a < r < b$, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \oint B \, ds = B \oint ds = B(2\pi r) = \mu_0 NI . \qquad (9.4.7)$$

The magnitude of the magnetic field is

$$B = \frac{\mu_0 NI}{2\pi r}, \qquad a < r < b, \qquad (9.4.8)$$

where r is the distance measured from the center of the toroid. Unlike the magnetic field of a solenoid, the magnetic field inside the toroid is non-uniform and decreases as $1/r$.

For the region $r < a$, there is no current enclosed in a circular Amperian loop of radius r, so the magnetic field is zero. In the region $r > b$, the enclosed current in a circular Amperian loop of radius r is $I_{enc} = NI - NI = 0$ because windings cut through the loop in opposite directions. Therefore the magnetic field is zero for $r > b$.

9.5 Magnetic Field of a Dipole

Let a magnetic dipole moment vector $\vec{\boldsymbol{\mu}} = -\mu_z \hat{\mathbf{k}}$ be placed at the origin (*e.g.*, center of the Earth) in the yz plane. What is the magnetic field at a point (*e.g.*, MIT) a distance r away from the origin?
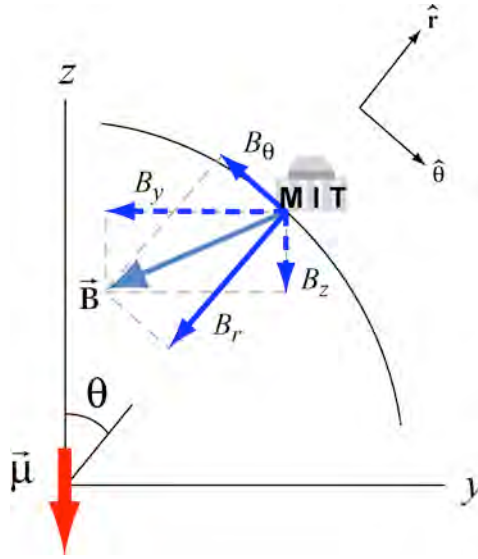
Figure 9.5.1 Earth's magnetic field components

In Figure 9.5.1 we show the magnetic field at MIT due to the dipole. The *y*- and *z*-components of the magnetic field are given by

$$B_y = -\frac{\mu_0}{4\pi}\frac{3\mu}{r^3}\sin\theta\cos\theta, \qquad B_z = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}(3\cos^2\theta - 1). \qquad (9.5.1)$$

(Readers are referred to Section 9.8 for the detail of the derivation.)

In spherical coordinates (r, θ, ϕ), the radial and the polar components of the magnetic field can be written as

$$B_r = B_y\sin\theta + B_z\cos\theta = -\frac{\mu_0}{4\pi}\frac{2\mu}{r^3}\cos\theta,$$
$$B_\theta = B_y\cos\theta - B_z\sin\theta = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}\sin\theta. \qquad (9.5.2)$$

Thus, the magnetic field at MIT due to the dipole becomes

$$\vec{B} = B_\theta\hat{\boldsymbol{\theta}} + B_r\hat{\mathbf{r}} = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}(\sin\theta\hat{\boldsymbol{\theta}} + 2\cos\theta\hat{\mathbf{r}}). \qquad (9.5.3)$$

Notice the similarity between the above expression and the electric field due to an electric dipole $\vec{p}$ (see Solved Problem 2.14.4),

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(\sin\theta\hat{\boldsymbol{\theta}} + 2\cos\theta\hat{\mathbf{r}}).$$

The negative sign in Eq. (9.5.3) is due to the fact that the magnetic dipole points in the negative z-direction. In general, the magnetic field due to a dipole moment $\vec{\mu}$ can be written as

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{\mu} \cdot \hat{r})\hat{r} - \vec{\mu}}{r^3}.$$ (9.5.4)

The ratio of the radial and the polar components is given by

$$\frac{B_r}{B_\theta} = \frac{-\dfrac{\mu_0}{4\pi} \dfrac{2\mu}{r^3} \cos\theta}{-\dfrac{\mu_0}{4\pi} \dfrac{\mu}{r^3} \sin\theta} = 2\cot\theta.$$ (9.5.5)

9.5.1 Earth's Magnetic Field at MIT

The Earth's field behaves as if there were a bar magnet in it. In Figure 9.5.2 an imaginary magnet is drawn inside the Earth oriented to produce a magnetic field like that of the Earth's magnetic field. Note the south pole of such a magnet is in the northern hemisphere in order to attract the north pole of a compass.
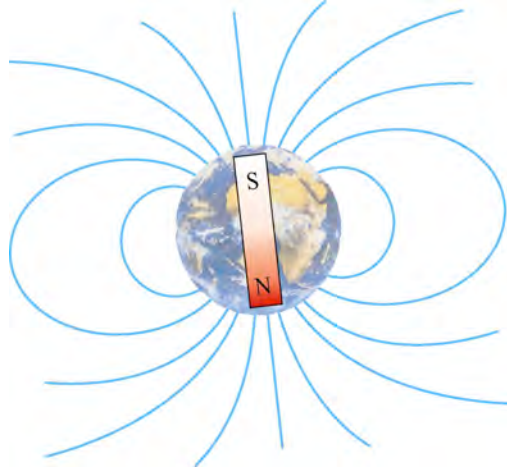
Figure 9.5.2 Magnetic field of the Earth

It is most natural to represent the location of a point P on the surface of the Earth using the spherical coordinates (r, θ, ϕ), where r is the distance from the center of the Earth, θ is the polar angle from the z-axis, with $0 \le \theta \le \pi$, and ϕ is the azimuthal angle in the xy-plane, measured from the x-axis, with $0 \le \phi \le 2\pi$ (See Figure 9.5.3.) With the distance fixed at $r = r_E$, the radius of the Earth, the point P is parameterized by the two angles θ and ϕ.
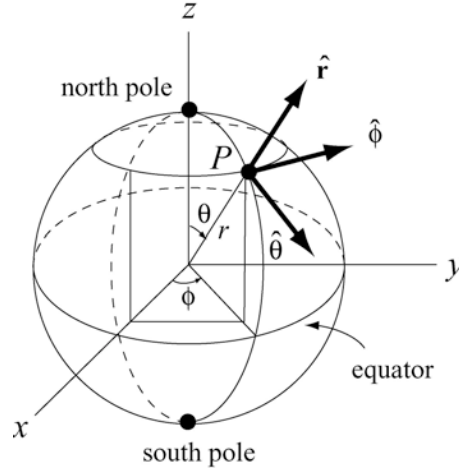
Figure 9.5.3 Locating a point P on the surface of the Earth using spherical coordinates.

In practice, two numbers describe a location on Earth: latitude and longitude. How are they related to θ and ϕ? The latitude of a point, denoted as λ, is a measure of the elevation from the plane of the equator. Thus, it is related to θ (commonly referred to as the co-latitude) by $\lambda = 90° - \theta$. Using this definition, the equator has latitude $0°$, and the north and the south poles have latitude $\pm 90°$, respectively.

The longitude of a location is simply represented by the azimuthal angle ϕ in the spherical coordinates. Lines of constant longitude are generally referred to as *meridians*. The value of longitude depends on where the counting begins. For historical reasons, the meridian passing through the Royal Astronomical Observatory in Greenwich, UK, is chosen as the "prime meridian" with zero longitude.

Let the z-axis be the Earth's rotation axis, and the x-axis passes through the prime meridian. The corresponding magnetic dipole moment of the Earth can be written as

$$
\begin{aligned}
\vec{\mu}_E &= \mu_E (\sin\theta_0 \cos\phi_0 \,\hat{\mathbf{i}} + \sin\theta_0 \sin\phi_0 \,\hat{\mathbf{j}} + \cos\theta_0 \,\hat{\mathbf{k}}) \\
&= \mu_E (-0.062\,\hat{\mathbf{i}} + 0.18\,\hat{\mathbf{j}} - 0.98\,\hat{\mathbf{k}}),
\end{aligned}
\tag{9.5.6}
$$

where $\mu_E = 7.79 \times 10^{22}\,\text{A} \cdot \text{m}^2$, and we have used $(\theta_0, \phi_0) = (169°, 109°)$. The expression shows that $\vec{\mu}_E$ has non-vanishing components in all three directions in the Cartesian coordinates.

On the other hand, the location of MIT is 42°N for the latitude and 71°W for the longitude ($42°$ north of the equator, and $71°$ west of the prime meridian), which means that $\theta_m = 90° - 42° = 48°$, and $\phi_m = 360° - 71° = 289°$. Thus, the position of MIT can be described by the vector

$$\vec{\mathbf{r}}_{\text{MIT}} = r_E(\sin\theta_m \cos\phi_m \,\hat{\mathbf{i}} + \sin\theta_m \sin\phi_m \,\hat{\mathbf{j}} + \cos\theta_m \,\hat{\mathbf{k}})$$
$$= r_E(0.24\,\hat{\mathbf{i}} - 0.70\,\hat{\mathbf{j}} + 0.67\,\hat{\mathbf{k}}). \tag{9.5.7}$$

The angle between $-\vec{\mathbf{\mu}}_E$ and $\vec{\mathbf{r}}_{\text{MIT}}$ is given by

$$\theta_{ME} = \cos^{-1}\left(\frac{-\vec{\mathbf{r}}_{\text{MIT}} \cdot \vec{\mathbf{\mu}}_E}{|\vec{\mathbf{r}}_{\text{MIT}}||-\vec{\mathbf{\mu}}_E|}\right) = \cos^{-1}(0.80) = 37°. \tag{9.5.8}$$

Note that the polar angle θ is defined as $\theta = \cos^{-1}(\hat{\mathbf{r}} \cdot \hat{\mathbf{k}})$, the inverse of cosine of the dot product between a unit vector $\hat{\mathbf{r}}$ for the position, and a unit vector $+\hat{\mathbf{k}}$ in the positive z-direction, as indicated in Figure 9.6.1. Thus, if we measure the ratio of the radial to the polar component of the Earth's magnetic field at MIT, the result would be

$$\boxed{\frac{B_r}{B_\theta} = 2\cot 37° \approx 2.65}. \tag{9.5.9}$$

The positive radial (vertical) direction is chosen to point outward and the positive polar (horizontal) direction points towards the equator.

9.6 Magnetic Materials

The introduction of material media into the study of magnetism has very different consequences as compared to the introduction of dielectric material media into the study of electrostatics. When we dealt with dielectric materials in electrostatics, their effect was *always* to reduce $\vec{\mathbf{E}}$ below what it would otherwise be, for a given amount of "free" electric charge. In contrast, when we deal with magnetic materials, their effect can be one of the following:

(i) reduce $\vec{\mathbf{B}}$ below what it would otherwise be, for the same amount of "free" electric current (*diamagnetic* materials);

(ii) increase $\vec{\mathbf{B}}$ a little above what it would otherwise be (*paramagnetic* materials);

(iii) increase $\vec{\mathbf{B}}$ *a lot* above what it would otherwise be (*ferromagnetic* materials).

Below we discuss how these effects arise.

9.6.1 Magnetization

Magnetic materials consist of many permanent or induced magnetic dipoles. One of the concepts crucial to the understanding of magnetic materials is the average magnetic field produced by many magnetic dipoles that are all aligned.
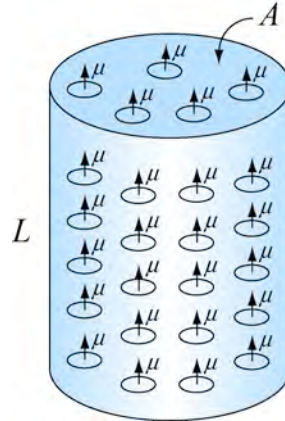
Figure 9.6.1 A cylinder with N magnetic dipole moments

Suppose we have a piece of material in the form of a long cylinder with area A and height L, and that it consists of N magnetic dipoles, each with magnetic dipole moment $\vec{\mu}$, of magnitude μ, spread uniformly throughout the volume of the cylinder, as shown in Figure 9.6.1. We also assume that all of the magnetic dipole moments $\vec{\mu}$ are aligned with the axis of the cylinder. In the absence of any external magnetic field, what is the average magnetic field due to these dipoles alone?

To answer this question, we note that each magnetic dipole has its own magnetic field associated with it. Let's define the magnetization vector $\vec{\mathbf{M}}$ to be the net magnetic dipole moment vector per unit volume,

$$\vec{\mathbf{M}} = \frac{1}{V} \sum_i \vec{\mu},$$
(9.6.1)

where V is the volume. In the case of the cylinder, where all the dipoles are aligned, the magnitude of $\vec{\mathbf{M}}$ is simply $M = N\mu / AL$. What is the average magnetic field produced by all the dipoles in the cylinder?
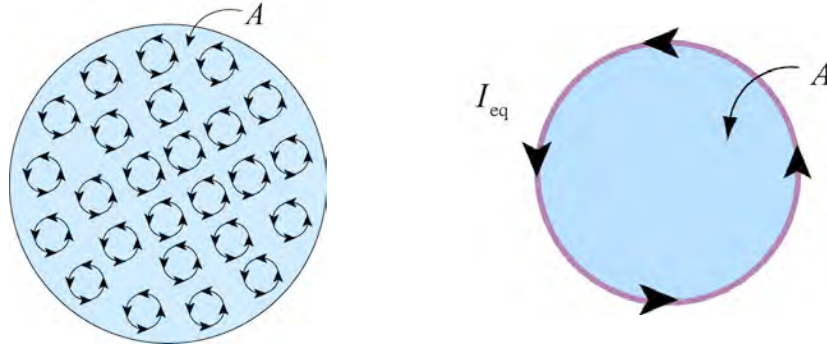
Figure 9.6.2 (a) Top view of the cylinder containing magnetic dipole moments. (b) The equivalent current.

Figure 9.6.2(a) depicts small current loops associated with the dipole moments and the direction of the currents, as seen from above. We see that in the interior, currents in a given direction are cancelled out by currents in the opposite direction in neighboring loops. The only place where cancellation does not take place is near the edge of the cylinder where there are no adjacent loops further out. Thus, the average current in the interior of the cylinder vanishes, whereas the sides of the cylinder appear to carry a current. The equivalent situation is shown in Figure 9.6.2(b), where there is an equivalent current I_{eq} on the sides.

The functional form of I_{eq} may be deduced by requiring that the magnetic dipole moment produced by I_{eq} is equal to the magnetic dipole moment of the system,

$$I_{eq}A = N\mu .$$
(9.6.2)

This condition requires that the equivalent current is given by

$$I_{eq} = \frac{N\mu}{A} .$$
(9.6.3)

Next, let's calculate the magnetic field produced by I_{eq}. With I_{eq} running on the sides, the equivalent configuration is identical to a solenoid carrying a surface current K. The two quantities are related by

$$K = \frac{I_{eq}}{L} = \frac{N\mu}{AL} = M$$
(9.6.4)

Thus, we see that the surface current K is equal to the magnetization M, which is the average magnetic dipole moment per unit volume. The average magnetic field inside the material produced by the equivalent current system is given by (see Section 9.4)

$$B_M = \mu_0 K = \mu_0 M . \tag{9.6.5}$$

Because the direction of this magnetic field is in the *same* direction as $\vec{\mathbf{M}}$, the above expression may be written in vector notation as

$$\vec{\mathbf{B}}_M = \mu_0 \vec{\mathbf{M}} . \tag{9.6.6}$$

This is exactly opposite from the situation with electric dipoles, in which the average electric field is anti-parallel to the direction of the electric dipoles themselves. The reason is that in the region interior to the current loop of a dipole, the magnetic field is in the *same* direction as the magnetic dipole vector. Therefore, it is not surprising that after a large-scale averaging, the average magnetic field also turns out to be parallel to the average magnetic dipole moment per unit volume. The magnetic field in Eq. (9.6.6) is the *average* field due to all the dipoles. A very different field is observed if we go close to any one of these little dipoles.

Let's now examine the properties of different magnetic materials

9.6.2 Paramagnetism

The atoms or molecules comprising paramagnetic materials have a permanent magnetic dipole moment. In the absence of any applied external magnetic field, the permanent magnetic dipoles in a paramagnetic material are randomly aligned. Thus, $\vec{\mathbf{M}} = \vec{\mathbf{0}}$ and the average magnetic field $\vec{\mathbf{B}}_M$ is also zero. However, when we place a paramagnetic material in an external field $\vec{\mathbf{B}}_0$, the dipoles experience a torque $\vec{\tau} = \vec{\mu} \times \vec{\mathbf{B}}_0$ that tends to align $\vec{\mu}$ with $\vec{\mathbf{B}}_0$, thereby producing a net magnetization $\vec{\mathbf{M}}$ parallel to $\vec{\mathbf{B}}_0$. Since $\vec{\mathbf{B}}_M$ is parallel to $\vec{\mathbf{B}}_0$, it will tend to *enhance* $\vec{\mathbf{B}}_0$. The total magnetic field $\vec{\mathbf{B}}$ is the sum of these two fields,

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \vec{\mathbf{B}}_M = \vec{\mathbf{B}}_0 + \mu_0 \vec{\mathbf{M}} . \tag{9.6.7}$$

This is a different result than in the case of dielectric materials. In both cases, the torque on the dipoles causes alignment of the dipole vector parallel to the external field. However, in the paramagnetic case, that alignment *enhances* the external magnetic field, whereas in the dielectric case it *reduces* the external electric field. In most paramagnetic substances, the magnetization $\vec{\mathbf{M}}$ is not only in the same direction as $\vec{\mathbf{B}}_0$, but also linearly proportional to $\vec{\mathbf{B}}_0$. This is plausible because without the external field $\vec{\mathbf{B}}_0$ there would be no alignment of dipoles and hence no magnetization $\vec{\mathbf{M}}$. The linear relation between $\vec{\mathbf{M}}$ and $\vec{\mathbf{B}}_0$ is expressed as

$$\vec{\mathbf{M}} = \chi_m \frac{\vec{\mathbf{B}}_0}{\mu_0} , \tag{9.6.8}$$

where χ_m is a dimensionless quantity called the *magnetic susceptibility*. Eq. (9.6.7) can then be written as

$$\vec{B} = (1 + \chi_m)\vec{B}_0 = \kappa_m\vec{B}_0, \qquad (9.6.9)$$

where

$$\kappa_m = 1 + \chi_m \qquad (9.6.10)$$

is called the *relative permeability* of the material. For paramagnetic substances, $\kappa_m > 1$, or equivalently, $\chi_m > 0$, although χ_m is usually on the order of 10^{-6} to 10^{-3}. The *magnetic permeability* μ_m of a material is defined as

$$\mu_m = (1 + \chi_m)\mu_0 = \kappa_m\mu_0. \qquad (9.6.11)$$

Paramagnetic materials have $\mu_m > \mu_0$.

9.6.3 Diamagnetism

In the case of magnetic materials where there are no permanent magnetic dipoles, the presence of an external field $\vec{B}_0$ will induce magnetic dipole moments in the atoms or molecules. However, these induced magnetic dipoles are anti-parallel to $\vec{B}_0$, leading to a magnetization $\vec{M}$, and average field $\vec{B}_M$ that is anti-parallel to $\vec{B}_0$, and therefore a *reduction* in the total magnetic field strength. For diamagnetic materials, we can still define the magnetic permeability, as in equation Eq. (9.6.11), although for diamagnetic materials $\kappa_m < 1$, hence $\chi_m < 0$, although χ_m is usually on the order of -10^{-5} to -10^{-9}. Diamagnetic materials have $\mu_m < \mu_0$.

9.6.4 Ferromagnetism

In ferromagnetic materials, there is a strong interaction between neighboring atomic dipole moments. Ferromagnetic materials are made up of small patches called *domains*, as illustrated in Figure 9.6.3(a). An externally applied magnetic field $\vec{B}_0$ will tend to line up those magnetic dipoles parallel to the external field, as shown in Figure 9.6.3(b). The strong interaction between neighboring atomic dipole moments causes a *much stronger* alignment of the magnetic dipoles than in paramagnetic materials.
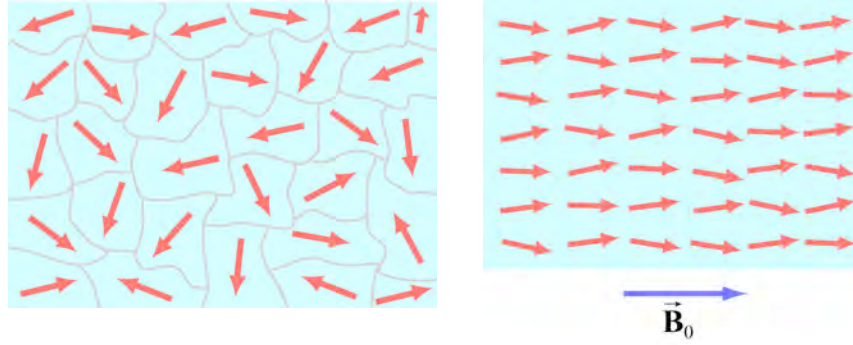
Figure 9.6.3 (a) Ferromagnetic domains. (b) Alignment of magnetic moments in the direction of the external field $\vec{B}_0$.

The enhancement of the applied external field can be considerable, with the total magnetic field inside a ferromagnet 10^3 or 10^4 times greater than the applied field.
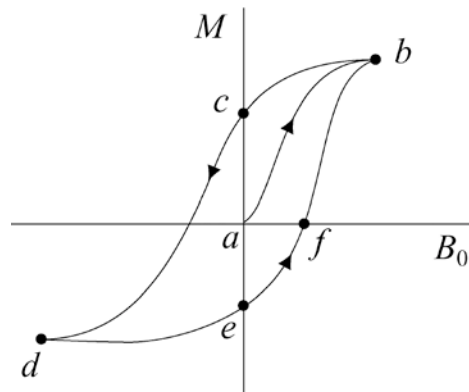
Figure 9.6.4 Plot of magnetization M vs. external field B_0 illustrating a hysteresis curve.

The permeability κ_m of a ferromagnetic material is not a constant, since neither the total field $\vec{B}$ nor the magnetization $\vec{M}$ increases linearly with $\vec{B}_0$. In fact the relationship between $\vec{M}$ and $\vec{B}_0$ is not unique, but dependent on the previous history of the material. The phenomenon is known as *hysteresis*. The variation of $\vec{M}$ as a function of the externally applied field $\vec{B}_0$ is shown in Figure 9.6.4. The loop *abcdef* is called a *hysteresis curve*.

Moreover, in ferromagnets, the strong interaction between neighboring atomic dipole moments can keep those dipole moments aligned, even when the external magnet field is reduced to zero. And these aligned dipoles can thus produce a strong magnetic field, all by themselves, without the necessity of an external magnetic field. This is the origin of permanent magnets. To see how strong such magnets can be, consider the fact that magnetic dipole moments of atoms typically have magnitudes of the order of 10^{-23} $A \cdot m^2$.

Typical atomic densities are 10^{29} atoms/m^3. If all these dipole moments were all aligned, then we would get a magnetization of order

$$M \sim (10^{-23} \text{ A} \cdot \text{m}^2)(10^{29} \text{ atoms/m}^3) \sim 10^6 \text{ A/m} \qquad (9.6.12)$$

The magnetization corresponds to values of $\vec{\mathbf{B}}_M = \mu_0 \vec{\mathbf{M}}$ of order 1 tesla, or 10,000 Gauss, just due to the atomic currents alone. This is how we get permanent magnets with fields of order 2000 Gauss.

9.7   Summary

- **Biot-Savart law** states that the magnetic field $d\vec{\mathbf{B}}$ at a point due to a length element $d\vec{\mathbf{s}}$ carrying a steady current I and located at $\vec{\mathbf{r}}$ away is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \frac{I \, d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2},$$

  where $r = |\vec{\mathbf{r}}|$ and $\mu_0 = 4\pi \times 10^{-7}$ T·m/A is the permeability of free space.

- The magnitude of the magnetic field at a distance r away from an infinitely long straight wire carrying a current I is

$$B = \frac{\mu_0 I}{2\pi r}.$$

- The magnitude of the magnetic force F_B between two straight wires of length s carrying steady current of I_1 and I_2 and separated by a distance r is

$$F_B = \frac{\mu_0 I_1 I_2 s}{2\pi r}$$

- **Ampere's law** states that the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total steady current passing through any surface that is bounded by the close loop,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}.$$

- The magnetic field inside a **toroid** which has N closely spaced of wire carrying a current I is given by

$$B = \frac{\mu_0 NI}{2\pi r}$$

where r is the distance from the center of the toroid.

- The magnetic field inside a **solenoid** which has N closely spaced of wire carrying current I in a length of l is given by

$$B = \mu_0 \frac{N}{l} I = \mu_0 n I \,,$$

where n is the number of number of turns per unit length.

- The properties of magnetic materials are as follows:
-

Materials	Magnetic susceptibility χ_m	Relative permeability $\kappa_m = 1 + \chi_m$	Magnetic permeability $\mu_m = \kappa_m \mu_0$
Diamagnetic	$-10^{-5} \sim -10^{-9}$	$\kappa_m < 1$	$\mu_m < \mu_0$
Paramagnetic	$10^{-5} \sim 10^{-3}$	$\kappa_m > 1$	$\mu_m > \mu_0$
Ferromagnetic	$\chi \gg 1$	$\kappa \gg 1$	$\mu \gg \mu_0$

9.8 Appendix 1: Magnetic Field off the Symmetry Axis of a Current Loop

In Example 9.2 we calculated the magnetic field due to a circular loop of radius R lying in the xy-plane and carrying a steady current I, at a point P along the axis of symmetry. Let's see how the same technique can be extended to calculating the field at a point off the axis of symmetry in the yz-plane.
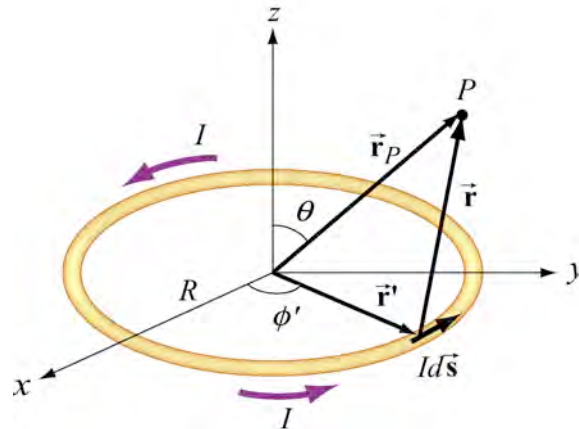
Figure 9.8.1 Calculating the magnetic field off the symmetry axis of a current loop.

The differential current element, as in Example 9.1, is

$$Id\vec{\mathbf{s}} = R\,d\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}).$$

Its position is described by $\vec{\mathbf{r}}' = R(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$. The field point P now lies in the yz-plane with $\vec{\mathbf{r}}_P = y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, as shown in Figure 9.8.1. The corresponding relative position vector is

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}' = -R\cos\phi'\hat{\mathbf{i}} + (y - R\sin\phi')\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad (9.8.1)$$

with a magnitude

$$r = |\vec{\mathbf{r}}| = \sqrt{(-R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2} = \sqrt{R^2 + y^2 + z^2 - 2yR\sin\phi}. \qquad (9.8.2)$$

The unit vector $\hat{\mathbf{r}}$ is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|},$$

pointing from $I\,d\vec{\mathbf{s}}$ to P. The cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ can be simplified as

$$\begin{aligned} d\vec{\mathbf{s}} \times \hat{\mathbf{r}} &= R\,d\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) \times [-R\cos\phi'\hat{\mathbf{i}} + (y - R\sin\phi')\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \\ &= R\,d\phi'[z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + (R - y\sin\phi')\hat{\mathbf{k}}]. \end{aligned} \qquad (9.8.3)$$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}} \times \vec{\mathbf{r}}}{r^3} = \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + (R - y\sin\phi')\hat{\mathbf{k}}}{(R^2 + y^2 + z^2 - 2yR\sin\phi')^{3/2}}d\phi'. \qquad (9.8.4)$$

Thus, the magnetic field at P is

$$\vec{\mathbf{B}}(0, y, z) = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + (R - y\sin\phi')\hat{\mathbf{k}}}{(R^2 + y^2 + z^2 - 2yR\sin\phi')^{3/2}}d\phi'. \qquad (9.8.5)$$

The x-component of $\vec{\mathbf{B}}$ can be readily shown to be zero,

$$B_x = \frac{\mu_0 IRz}{4\pi} \int_0^{2\pi} \frac{\cos\phi' d\phi'}{(R^2 + y^2 + z^2 - 2yR\sin\phi')^{3/2}} = 0, \qquad (9.8.6)$$

by making a change of variable $w = R^2 + y^2 + z^2 - 2yR\sin\phi'$, followed by a straightforward integration. One may also invoke symmetry arguments to verify that B_x must vanish; namely, the contribution at ϕ' is cancelled by the contribution at $\pi - \phi'$. The y and the z components of $\vec{B}$ are

$$B_y = \frac{\mu_0 IRz}{4\pi} \int_0^{2\pi} \frac{\sin\phi' d\phi'}{(R^2 + y^2 + z^2 - 2yR\sin\phi')^{3/2}},$$

$$B_z = \frac{\mu_0 IR}{4\pi} \int_0^{2\pi} \frac{(R - y\sin\phi')d\phi'}{(R^2 + y^2 + z^2 - 2yR\sin\phi')^{3/2}}. \qquad (9.8.7)$$

These integrals are *elliptic integrals* that can be evaluated numerically.

In the limit $y = 0$, the field point P is located along the z-axis, and we recover the results obtained in Example 9.2,

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \int_0^{2\pi} \sin\phi' d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \cos\phi' \Big|_0^{2\pi} = 0$$

$$B_z = \frac{\mu_0}{4\pi} \frac{IR^2}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi} \frac{2\pi IR^2}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2 + z^2)^{3/2}}. \qquad (9.8.8)$$

Now, let's consider the "point-dipole" limit where $R \ll (y^2 + z^2)^{1/2} = r$, corresponding to the case when the characteristic dimension of the current source is much smaller compared to the distance to the field point. In this limit, the denominator in the integrand can be expanded as

$$(R^2 + y^2 + z^2 - 2yR\sin\phi')^{-3/2} = \frac{1}{r^3}\left[1 + \frac{R^2 - 2yR\sin\phi'}{r^2}\right]^{-3/2}$$

$$= \frac{1}{r^3}\left[1 - \frac{3}{2}\left(\frac{R^2 - 2yR\sin\phi'}{r^2}\right) + \ldots\right]. \qquad (9.8.9)$$

The y-component of the magnetic field is then

$$B_y \approx \frac{\mu_0 I}{4\pi} \frac{Rz}{r^3} \int_0^{2\pi} \left[1 - \frac{3}{2}\left(\frac{R^2 - 2yR\sin\phi'}{r^2} \right) \right] \sin\phi'\,d\phi'$$

$$= \frac{\mu_0 I}{4\pi} \frac{3R^2 yz}{r^5} \int_0^{2\pi} \sin^2\phi'\,d\phi' = \frac{\mu_0 I}{4\pi} \frac{3\pi R^2 yz}{r^5}. \qquad (9.8.10)$$

The *z*-component of the magnetic field is then

$$B_z \approx \frac{\mu_0 I}{4\pi} \frac{R}{r^3} \int_0^{2\pi} \left[1 - \frac{3}{2}\left(\frac{R^2 - 2yR\sin\phi'}{r^2} \right) \right] (R - y\sin\phi')\,d\phi'$$

$$= \frac{\mu_0 I}{4\pi} \frac{R}{r^3} \int_0^{2\pi} \left[\left(R - \frac{3R^3}{2r^2} \right) - \left(1 - \frac{9R^2}{2r^2} \right)\sin\phi' - \frac{3Ry^2}{r^2}\sin^2\phi' \right] d\phi'$$

$$= \frac{\mu_0 I}{4\pi} \frac{R}{r^3} \left[2\pi\left(R - \frac{3R^3}{2r^2} \right) - \frac{3\pi Ry^2}{r^2} \right] \qquad (9.8.11)$$

$$= \frac{\mu_0 I}{4\pi} \frac{\pi R^2}{r^3} \left[2 - \frac{3y^2}{r^2} + \text{higher order terms} \right].$$

The quantity $I(\pi R^2)$ may be identified as the magnitude of the magnetic dipole moment $\mu = IA$, where $A = \pi R^2$ is the area of the loop. Using spherical coordinates with $y = r\sin\theta$ and $z = r\cos\theta$, the above expressions may be rewritten as

$$B_y = \frac{\mu_0 (I\pi R^2)}{4\pi} \frac{3(r\sin\theta)(r\cos\theta)}{r^5} = \frac{\mu_0}{4\pi} \frac{3\mu\sin\theta\cos\theta}{r^3}, \qquad (9.8.12)$$

and

$$B_z = \frac{\mu_0}{4\pi} \frac{(I\pi R^2)}{r^3} \left(2 - \frac{3r^2\sin^2\theta}{r^2} \right) = \frac{\mu_0}{4\pi} \frac{\mu}{r^3} (2 - 3\sin^2\theta) = \frac{\mu_0}{4\pi} \frac{\mu}{r^3} (3\cos^2\theta - 1). \qquad (9.8.13)$$

Thus, we see that the magnetic field at a point $r \gg R$ due to a current ring of radius R may be approximated by a small magnetic dipole moment placed at the origin (Figure 9.8.2).
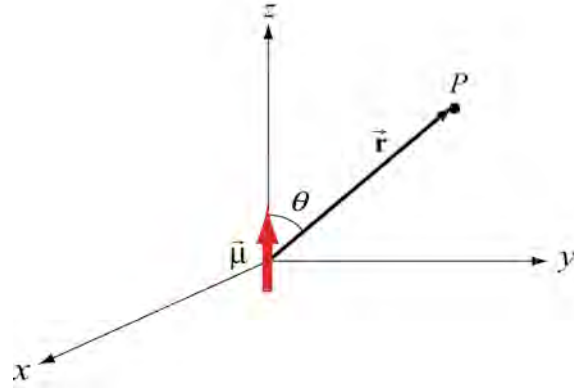
Figure 9.8.2 Magnetic dipole moment $\vec{\mu} = \mu\hat{k}$

The magnetic field lines due to a current loop and a dipole moment (small bar magnet) are depicted in Figure 9.8.3.

Figure 9.8.3 Magnetic field lines due to (a) a current loop, and (b) a small bar magnet.

The magnetic field at P can also be written in spherical coordinates as

$$\vec{B} = B_r\,\hat{r} + B_\theta\,\hat{\theta}. \tag{9.8.14}$$

The spherical components B_r and B_θ are related to the Cartesian components B_y and B_z by

$$B_r = B_y\sin\theta + B_z\cos\theta, \qquad B_\theta = B_y\cos\theta - B_z\sin\theta. \tag{9.8.15}$$

In addition, we have, for the unit vectors,

$$\hat{r} = \sin\theta\,\hat{j} + \cos\theta\,\hat{k}, \qquad \hat{\theta} = \cos\theta\,\hat{j} - \sin\theta\,\hat{k}. \tag{9.8.16}$$

Using the above relations, the spherical components may be written as

$$B_r = \frac{\mu_0 I R^2 \cos\theta}{4\pi} \int_0^{2\pi} \frac{d\phi'}{(R^2 + r^2 - 2rR\sin\theta\sin\phi')^{3/2}}, \qquad (9.8.17)$$

and

$$B_\theta(r,\theta) = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{(r\sin\phi' - R\sin\theta)\,d\phi'}{(R^2 + r^2 - 2rR\sin\theta\sin\phi')^{3/2}}. \qquad (9.8.18)$$

In the limit where $R < r$, we obtain

$$B_r \approx \frac{\mu_0 I R^2 \cos\theta}{4\pi r^3} \int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi} \frac{2\pi I R^2 \cos\theta}{r^3} = \frac{\mu_0}{4\pi} \frac{2\mu\cos\theta}{r^3}, \qquad (9.8.19)$$

and

$$
\begin{aligned}
B_\theta &= \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{(r\sin\phi' - R\sin\theta)\,d\phi'}{(R^2 + r^2 - 2rR\sin\theta\sin\phi')^{3/2}} \\
&\approx \frac{\mu_0 I R}{4\pi r^3} \int_0^{2\pi} \left[-R\sin\theta\left(1 - \frac{3R^2}{2r^2}\right) + \left(r - \frac{3R^2}{2r} - \frac{3R^2\sin^2\theta}{2r}\right)\sin\phi' + 3R\sin\theta\sin^2\phi' \right] d\phi' \\
&\approx \frac{\mu_0 I R}{4\pi r^3}\left(-2\pi R\sin\theta + 3\pi R\sin\theta\right) = \frac{\mu_0 (I\pi R^2)\sin\theta}{4\pi r^3} \\
&= \frac{\mu_0}{4\pi} \frac{\mu\sin\theta}{r^3}.
\end{aligned}
$$

$$(9.8.20)$$

9.9 Appendix 2: Helmholtz Coils

Consider two *N*-turn circular coils of radius *R*, each perpendicular to the axis of symmetry, with their centers located at $z = \pm l/2$. There is a steady current *I* in the same direction around each coil, as shown in Figure 9.9.1. Let's find the magnetic field $\vec{\mathbf{B}}$ on the axis at a distance *z* from the center of one coil.
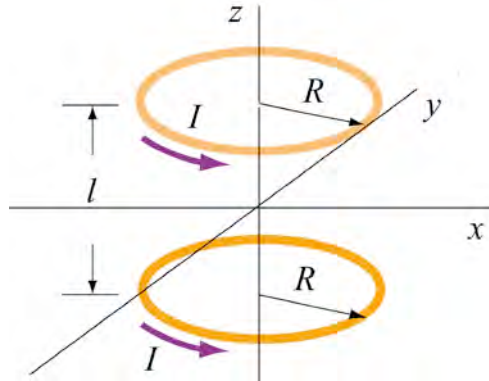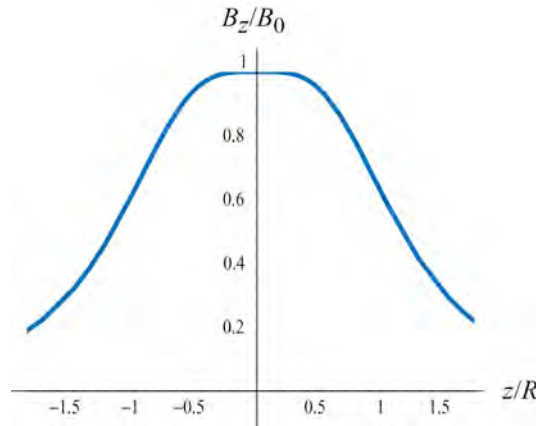

Figure 9.9.1 Helmholtz coils

Let's find the magnetic field on the z-axis at a point, a distance $z - l/2$ away from the center of one ring and a distance $z + l/2$ from the center of the other ring. Using the result shown in Example 9.2 for a single coil and applying the superposition principle, the magnetic field at $P(z,0)$ due to the two coils can be obtained as

$$B_z = B_{\text{top}} + B_{\text{bottom}} = \frac{\mu_0 NIR^2}{2}\left[\frac{1}{[(z-l/2)^2 + R^2]^{3/2}} + \frac{1}{[(z+l/2)^2 + R^2]^{3/2}}\right]. \qquad (9.9.1)$$

The magnetic field strength at $z = 0$ and $l = R$ is given by

$$B_0 = \frac{\mu_0 NI}{(5/4)^{3/2} R}. \qquad (9.9.2)$$

A plot of B_z / B_0 as a function of z / R is depicted in Figure 9.9.2.

Figure 9.9.2 Plot of magnetic field as a function of z/R for Helmholtz coils.

Let's analyze the properties of B_z in more detail. Differentiating B_z with respect to z, we obtain

$$\frac{dB_z}{dz} = \frac{\mu_0 NIR^2}{2}\left\{-\frac{3(z-l/2)}{[(z-l/2)^2 + R^2]^{5/2}} - \frac{3(z+l/2)}{[(z+l/2)^2 + R^2]^{5/2}}\right\}. \qquad (9.9.3)$$

One may readily show that at the midpoint, $z = 0$, the first derivative vanishes,

$$\left.\frac{dB}{dz}\right|_{z=0} = 0. \qquad (9.9.4)$$

The second derivative is

$$\frac{d^2B}{dz^2} = \frac{N\mu_0 IR^2}{2}\left\{ -\frac{3}{[(z-l/2)^2+R^2]^{5/2}} + \frac{15(z-l/2)^2}{[(z-l/2)^2+R^2]^{7/2}} \right.$$
$$\left. -\frac{3}{[(z+l/2)^2+R^2]^{5/2}} + \frac{15(z+l/2)^2}{[(z+l/2)^2+R^2]^{7/2}} \right\}.$$

(9.9.5)

At the midpoint $z=0$, the Eq. (9.9.5) simplifies to

$$\frac{d^2B}{dz^2}\bigg|_{z=0} = \frac{\mu_0 NI^2 R^2}{2}\left\{ -\frac{6}{[(l/2)^2+R^2]^{5/2}} + \frac{15l^2}{2[(l/2)^2+R^2]^{7/2}} \right\}$$
$$= -\frac{\mu_0 NI^2 R^2}{2}\frac{6(R^2-l^2)}{[(l/2)^2+R^2]^{7/2}}.$$

(9.9.6)

When the distance between centers of the two coils is equal to the radius of the coil, $l=R$, the second derivative of B_z vanishes at $z=0$. A configuration with $l=R$ is known as *Helmholtz coils*.

For small z, we may make a Taylor-series expansion of $B_z(z)$ about $z=0$,

$$B_z(z) = B_z(0) + \frac{dB_z}{dz}(0)z + \frac{1}{2!}\frac{d^2B_z}{dz^2}(0)z^2 + \cdots.$$

(9.9.7)

The fact that the first two derivatives vanish at $z=0$ indicates that the magnetic field is fairly uniform in the small z region. It turns out that the third derivative vanishes at $z=0$ as well.

Recall that the force experienced by a dipole in a magnetic field is $\vec{F}_B = \nabla(\vec{\mu}\cdot\vec{B})$. If we place a magnetic dipole $\vec{\mu} = \mu_z\,\hat{k}$ at $z=0$, the magnetic force acting on the dipole is

$$\vec{F}_B = \nabla(\mu_z B_z) = \mu_z \frac{dB_z}{dz}\hat{k},$$

(9.9.8)

which will be very small because the magnetic field is nearly uniform there.

9.9.1 Magnetic Field of the Helmholtz Coils Movie

The animation in Figure 9.9.3(a) shows the magnetic field of the Helmholtz coils. In this configuration the currents in the top and bottom coils are in the same direction, with their dipole moments aligned. The magnetic fields from the two coils add up to create a net field that is nearly uniform at the center of the coils. Since the distance between the coils is equal to the radius of the coils and remains unchanged, the force of attraction between

them creates a tension, and is illustrated by field lines stretching out to enclose both coils. When the distance between the coils is not fixed, as in the animation depicted in Figure 9.9.3(b), the two coils move toward each other due to their force of attraction. In this animation, the top loop has only half the current as the bottom loop. The field configuration is shown using the "iron filings" representation.


(a) http://youtu.be/DNbSICRSbeU   (b) http://youtu.be/rUL71B6gGd8
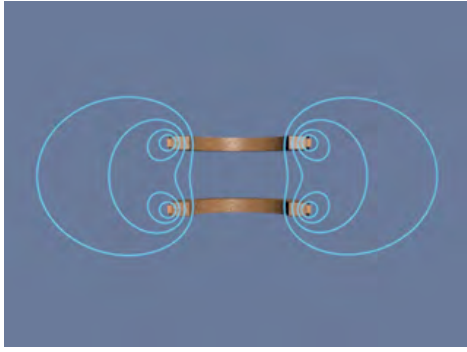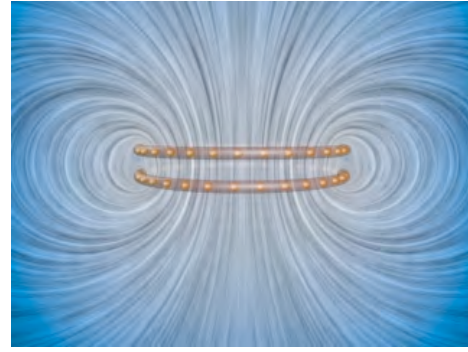
Figure 9.9.3 (a) Magnetic field of the Helmholtz coils where the distance between the coils is equal to the radius of the coil. (b) Two co-axial wire loops carrying current in the same sense are attracted to each other.

Next, let's consider the case where the currents in the loop are in the opposite directions, as shown in Figure 9.9.4.
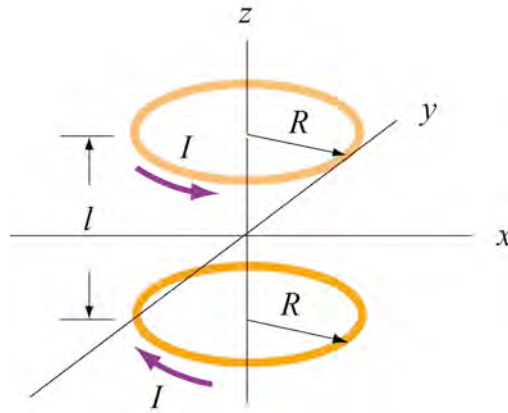
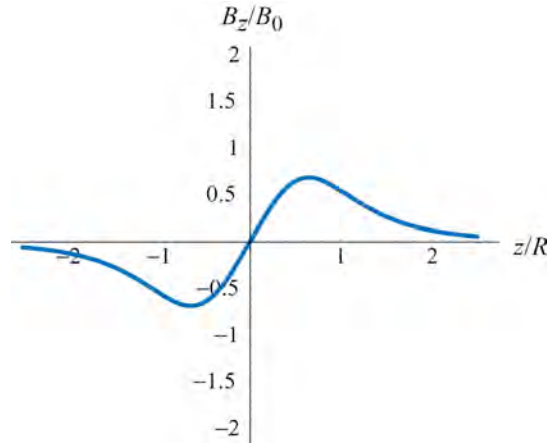**Figure 9.9.4** Two circular loops carrying currents in the opposite directions.

Again, by superposition principle, the magnetic field at a point $P(0,0,z)$ with $z > 0$ is

$$B_z = B_{1z} + B_{2z} = \frac{\mu_0 NIR^2}{2}\left[\frac{1}{[(z-l/2)^2 + R^2]^{3/2}} - \frac{1}{[(z+l/2)^2 + R^2]^{3/2}}\right]. \qquad (9.9.9)$$

A plot of B_z / B_0 with $B_0 = \mu_0 NI / 2R$ and $l = R$ is depicted in Figure 9.9.5.

Figure 9.9.5 Plot of magnetic field as a function of z/R for anti-Helmholtz coils.

Differentiating B_z with respect to z, we obtain

$$\frac{dB_z}{dz} = \frac{\mu_0 NIR^2}{2}\left\{-\frac{3(z-l/2)}{[(z-l/2)^2 + R^2]^{5/2}} + \frac{3(z+l/2)}{[(z+l/2)^2 + R^2]^{5/2}}\right\}. \qquad (9.9.10)$$

At the midpoint, $z = 0$, we have

$$\frac{dB_z}{dz}(0) = \frac{\mu_0 NIR^2}{2}\frac{3l}{[(l/2)^2 + R^2]^{5/2}} \neq 0. \qquad (9.9.11)$$

Thus, a magnetic dipole $\vec{\mu} = \mu_z\,\hat{\mathbf{k}}$ placed at $z = 0$ will experience a net force given by

$$\vec{\mathbf{F}}_B = \nabla(\vec{\mu}\cdot\vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z\frac{dB_z(0)}{dz}\hat{\mathbf{k}} = \frac{\mu_z\mu_0 NIR^2}{2}\frac{3l}{[(l/2)^2 + R^2]^{5/2}}\hat{\mathbf{k}} \quad . \qquad (9.9.12)$$

For $l = R$, the above expression simplifies to

$$\vec{\mathbf{F}}_B = \frac{3\mu_z\mu_0 NI}{2(5/4)^{5/2} R^2}\hat{\mathbf{k}}. \qquad (9.9.13)$$

9.9.2 Magnetic Field of Two Coils Carrying Opposite Currents Movie

The animation depicted in Figure 9.9.6 shows the magnetic field of two coils like the Helmholtz coils but with currents in the top and bottom coils in opposite directions. In this configuration, the magnetic dipole moments associated with each coil are anti-parallel.

(a) http://youtu.be/SD4fq8wPee0   (b) http://youtu.be/9Ug-VxtCsSw
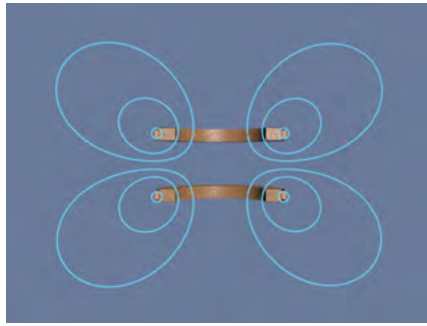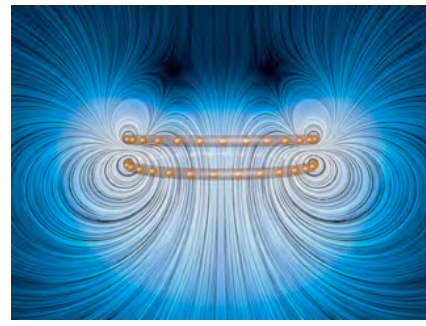
**Figure 9.9.6** (a) Magnetic field due to coils carrying currents in the opposite directions. (b) Two co-axial wire loops carrying current in the opposite sense repel each other. The field configurations here are shown using the "iron filings" representation. The bottom wire loop carries twice the amount of current as the top wire loop.

At the center of the coils along the axis of symmetry, the magnetic field is zero. With the distance between the two coils fixed, the repulsive force results in a pressure between them. This is illustrated by field lines that are compressed along the central horizontal axis between the coils.

9.9.3 Forces Between Coaxial Current-Carrying Wires Movie

http://youtu.be/2TWYhylF4K4
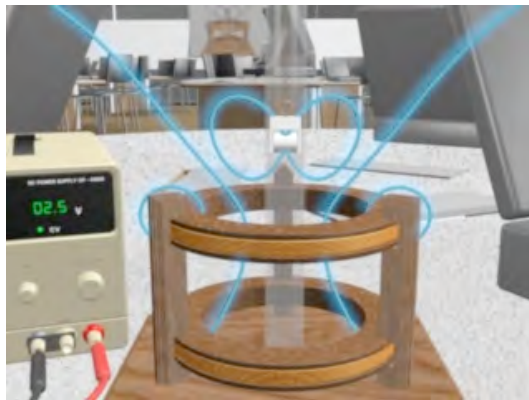
Figure 9.9.7 A magnet in the ***TeachSpin™*** Magnetic Force apparatus when the current in the top coil is counterclockwise as seen from the top.

Figure 9.9.7 shows the force of repulsion between the magnetic field of a permanent magnet and the field of a current-carrying ring in the *TeachSpin™* Magnetic Force apparatus. The magnet is constrained to have its north magnetic pole pointing downward, and the current in the top coil of the Magnetic Force apparatus is moving clockwise as seen from above. The net result is a repulsion of the magnet when the current in this direction is increased. The visualization shows the stresses transmitted by the fields to the magnet when the current in the upper coil is increased.

9.9.4 Magnet Oscillating Between Two Coils Movie

Figure 9.9.8 illustrates an animation in which the magnetic field of a permanent magnet suspended by a spring in the *TeachSpin™* apparatus (see *TeachSpin* visualization), plus the magnetic field due to current in the two coils (here we see a "cutaway" cross-section of the apparatus).

http://youtu.be/poChtIhICuQ
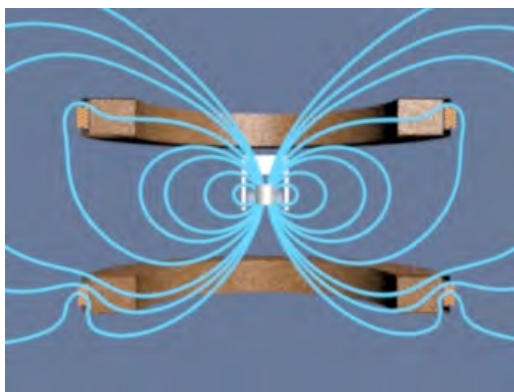
Figure 9.9.8 Magnet oscillating between two coils.

The magnet is fixed so that its north pole points upward, and the current in the two coils is sinusoidal and 180 degrees out of phase. When the effective dipole moment of the top coil points upwards, the dipole moment of the bottom coil points downwards. Thus, the magnet is attracted to the upper coil and repelled by the lower coil, causing it to move upwards. When the conditions are reversed during the second half of the cycle, the magnet moves downwards.

This process can also be described in terms of tension along, and pressure perpendicular to, the field lines of the resulting field. When the dipole moment of one of the coils is aligned with that of the magnet, there is a tension along the field lines as they attempt to "connect" the coil and magnet. Conversely, when their moments are anti-aligned, there is a pressure perpendicular to the field lines as they try to keep the coil and magnet apart.

9.9.5 Magnet Suspended Between Two Coils Movie

Figure 9.9.9 illustrates an animation in which the magnetic field of a permanent magnet suspended by a spring in the *TeachSpin™* apparatus, plus the magnetic field due to current in the two coils (here we see a "cutaway" cross-section of the apparatus). The magnet is fixed so that its north pole points upward, and the current in the two coils is sinusoidal and in phase. When the effective dipole moment of the top coil points upwards, the dipole moment of the bottom coil points upward as well. Thus, the magnet is attracted to both coils, and as a result feels no net force (although it does feel a torque, not shown here since the direction of the magnet is fixed to point upwards). When the dipole

moments are reversed during the second half of the cycle, the magnet is repelled by both coils, again resulting in no net force.

This process can also be described in terms of tension along, and pressure perpendicular to, the field lines of the resulting field. When the dipole moment of the coils is aligned with that of the magnet, there is a tension along the field lines as they are "pulled" from both sides. Conversely, when their moments are anti-aligned, there is a pressure perpendicular to the field lines as they are "squeezed" from both sides.

http://youtu.be/mHX7v0mgTak
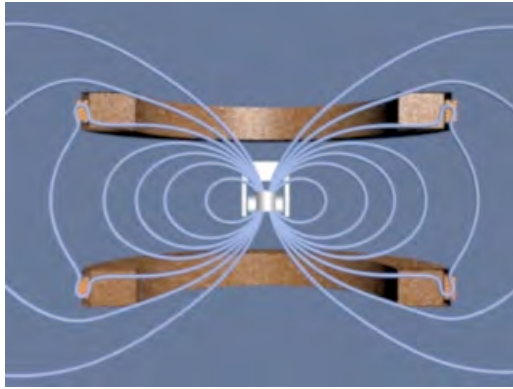
Figure 9.9.9 Magnet suspended between two coils carrying currents in the same direction.

9.10   Problem-Solving Strategies

In this chapter, we have seen how Biot-Savart and Ampere's laws can be used to calculate magnetic field due to a current source.

9.10.1 Biot-Savart Law:

The law states that the magnetic field at a point P due to a length element $d\vec{s}$ carrying a steady current I located at $\vec{r}$ away is given by

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \vec{r}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $I\,d\vec{s}$, and the vector $\vec{r}'$ describing the position of $I\,d\vec{s}$. The magnitude $r' = |\vec{r}'|$ is the distance between $I\,d\vec{s}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point P is the point in space where we are trying to calculate the magnetic field due to the current. Using the same coordinate system, write down the position vector $\vec{\mathbf{r}}_P$ for the field point P. The quantity $r_P = |\vec{\mathbf{r}}_P|$ is the distance between the origin and P.

(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|} ,$$

where $r = |\vec{\mathbf{r}}| = |\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|$ is the distance between the source and the field point P.
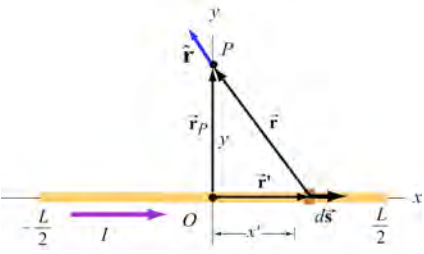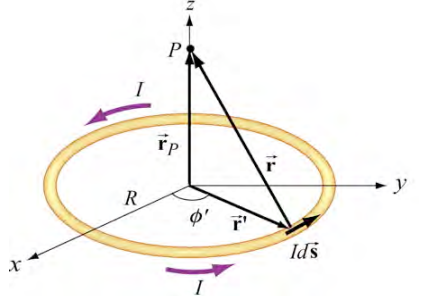
(4) Calculate the vector product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ or $d\vec{\mathbf{s}} \times \vec{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field $\vec{\mathbf{B}}$, according to the Biot-Savart law.

(5) Substitute the expressions obtained to $d\vec{\mathbf{B}}$ and simplify as much as possible.

(6) Complete the integration to obtain $\vec{\mathbf{B}}$ if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

Below we illustrate how these steps are executed for a current-carrying wire of length L and a loop of radius R.

Current distribution	Finite wire of length L	Circular loop of radius R
Figure		
(1) Source point	$\vec{\mathbf{r}}'=x'\hat{\mathbf{i}}$ $d\vec{\mathbf{s}}=(d\vec{\mathbf{r}}'/dx')dx'=dx'\hat{\mathbf{i}}$	$\vec{\mathbf{r}}'=R(\cos\phi'\hat{\mathbf{i}}+\sin\phi'\hat{\mathbf{j}})$ $d\vec{\mathbf{s}}=(d\vec{\mathbf{r}}'/d\phi')d\phi'=Rd\phi'(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}})$
(2) Field point P	$\vec{\mathbf{r}}_P=y\hat{\mathbf{j}}$	$\vec{\mathbf{r}}_P=z\hat{\mathbf{k}}$
(3) Relative position vector $\vec{\mathbf{r}}=\vec{\mathbf{r}}_P-\vec{\mathbf{r}}'$	$\vec{\mathbf{r}}=y\hat{\mathbf{j}}-x'\hat{\mathbf{i}}$ $r=\lvert\vec{\mathbf{r}}\rvert=\sqrt{x'^2+y^2}$ $\hat{\mathbf{r}}=\dfrac{y\hat{\mathbf{j}}-x'\hat{\mathbf{i}}}{\sqrt{x'^2+y^2}}$	$\vec{\mathbf{r}}=-R\cos\phi'\hat{\mathbf{i}}-R\sin\phi'\hat{\mathbf{j}}+z\hat{\mathbf{k}}$ $r=\lvert\vec{\mathbf{r}}\rvert=\sqrt{R^2+z^2}$ $\hat{\mathbf{r}}=\dfrac{-R\cos\phi'\hat{\mathbf{i}}-R\sin\phi'\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{\sqrt{R^2+z^2}}$
(4) The cross product $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$	$d\vec{\mathbf{s}}\times\hat{\mathbf{r}}=\dfrac{y\,dx'\hat{\mathbf{k}}}{\sqrt{y^2+x'^2}}$	$d\vec{\mathbf{s}}\times\hat{\mathbf{r}}=\dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}}+z\sin\phi'\hat{\mathbf{j}}+R\hat{\mathbf{k}})}{\sqrt{R^2+z^2}}$
(5) Rewrite $d\vec{\mathbf{B}}$	$d\vec{\mathbf{B}}=\dfrac{\mu_0 I}{4\pi}\dfrac{y\,dx'\hat{\mathbf{k}}}{(y^2+x'^2)^{3/2}}$	$d\vec{\mathbf{B}}=\dfrac{\mu_0 I}{4\pi}\dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}}+z\sin\phi'\hat{\mathbf{j}}+R\hat{\mathbf{k}})}{(R^2+z^2)^{3/2}}$
(6) Integrate to get $\vec{\mathbf{B}}$	$B_x=0$ $B_y=0$ $B_z=\dfrac{\mu_0 Iy}{4\pi}\displaystyle\int_{-L/2}^{L/2}\dfrac{dx'}{(y^2+x'^2)^{3/2}}$ $=\dfrac{\mu_0 I}{4\pi}\dfrac{L}{y\sqrt{y^2+(L/2)^2}}$	$B_x=\dfrac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}\cos\phi'\,d\phi'=0$ $B_y=\dfrac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}\sin\phi'\,d\phi'=0$ $B_z=\dfrac{\mu_0 IR^2}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}d\phi'=\dfrac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$

9.10.2 Ampere's law:
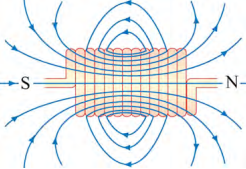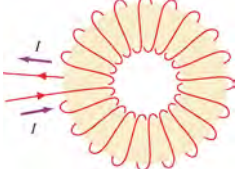
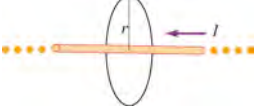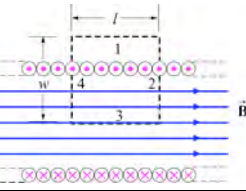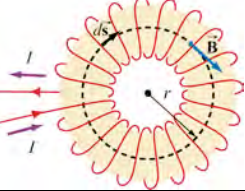Ampere's law states that the line integral of $\vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop,

$$\oint\vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}=\mu_0 I_{enc}\,.$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

(1) Draw an Amperian loop using symmetry arguments.

(2) Find the current enclosed by the Amperian loop.

(3) Calculate the line integral $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around the closed loop.

(4) Equate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ with $\mu_0 I_{enc}$ and solve for $\vec{\mathbf{B}}$.

Below we summarize how the methodology can be applied to calculate the magnetic field for an infinite wire, an ideal solenoid and a toroid.

System	Infinite wire	Ideal solenoid	Toroid
Figure			
(1) Draw the Amperian loop			
(2) Find the current enclosed by the Amperian loop	$I_{enc} = I$	$I_{enc} = NI$	$I_{enc} = NI$
(3) Calculate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ along the loop	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl$	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$
(4) Equate $\mu_0 I_{enc}$ with $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ to obtain $\vec{\mathbf{B}}$	$B = \dfrac{\mu_0 I}{2\pi r}$	$B = \dfrac{\mu_0 NI}{l} = \mu_0 nI$	$B = \dfrac{\mu_0 NI}{2\pi r}$

9.11   Solved Problems

9.11.1 Magnetic Field of a Straight Wire

Consider a straight wire of length L carrying a current I along the positive x-direction, as shown in Figure 9.11.1 (ignore the return path of the current or the source for the current.) What is the magnetic field at an arbitrary point P on the xy-plane?
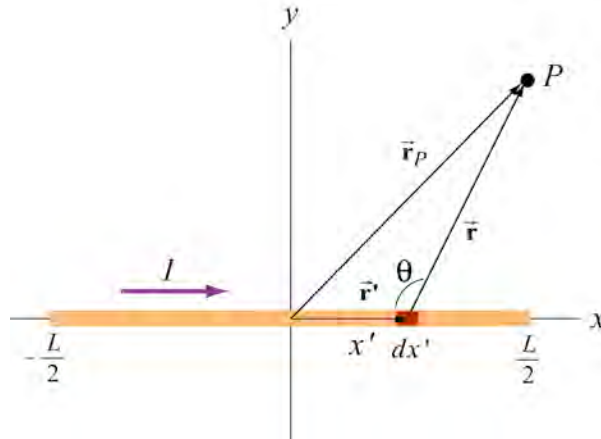
Figure 9.11.1 A finite straight wire carrying a current I.

Solution: The problem is very similar to Example 9.1. However, now the field point is an arbitrary point in the xy-plane. We solve the problem using the methodology outlined in Section 9.10.

(1) <u>Source point:</u> From Figure 9.10.1, we see that the infinitesimal length dx' described by the position vector $\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$ constitutes a current source $I\,d\vec{\mathbf{s}} = (I dx')\hat{\mathbf{i}}$.

(2) <u>Field point:</u> As can be seen from Figure 9.10.1, the position vector for the field point P is $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

(3) <u>Relative position vector:</u> The relative position vector from the source to P is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}' = (x - x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, with magnitude $r = |\vec{\mathbf{r}}_P| = |\vec{\mathbf{r}} - \vec{\mathbf{r}}'| = [(x - x')^2 + y^2]^{1/2}$. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|} = \frac{(x - x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{[(x - x')^2 + y^2]^{1/2}}.$$

(4) <u>Simplifying the vector product:</u> The vector product $d\vec{\mathbf{s}} \times \vec{\mathbf{r}}$ can be simplified as

$$(dx'\hat{\mathbf{i}}) \times [(x - x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}] = y\,dx'\hat{\mathbf{k}},$$

9-51

where we have used $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$ and $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$.

(5) <u>Writing down $d\vec{\mathbf{B}}$</u>: Using the Biot-Savart law, the infinitesimal contribution due to $I\,d\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \vec{\mathbf{r}}}{r^3} = \frac{\mu_0 I}{4\pi} \frac{y\,dx'}{[(x-x')^2 + y^2]^{3/2}} \hat{\mathbf{k}}.$$ (9.11.1)

The direction of the magnetic field is in the $+\hat{\mathbf{k}}$ direction.

(6) <u>Carrying out the integration to obtain $\vec{\mathbf{B}}$</u>: The total magnetic field at P can then be obtained by integrating over the entire length of the wire,

$$\vec{\mathbf{B}} = \int_{\text{wire}} d\vec{\mathbf{B}} = \int_{-L/2}^{L/2} \frac{\mu_0 I y\,dx'}{4\pi[(x-x')^2 + y^2]^{3/2}} \hat{\mathbf{k}} = -\frac{\mu_0 I}{4\pi y} \frac{(x-x')}{\sqrt{(x-x')^2 + y^2}} \Bigg|_{-L/2}^{L/2} \hat{\mathbf{k}}$$

$$= -\frac{\mu_0 I}{4\pi y} \left[\frac{(x-L/2)}{\sqrt{(x-L/2)^2 + y^2}} - \frac{(x+L/2)}{\sqrt{(x+L/2)^2 + y^2}} \right] \hat{\mathbf{k}}.$$ (9.11.2)

Let's consider the following limits:

(i) $x = 0$: In this case, the field point P is at $(x, y) = (0, y)$ on the y-axis. The magnetic field becomes

$$\vec{\mathbf{B}} = -\frac{\mu_0 I}{4\pi y} \left[\frac{-L/2}{\sqrt{(-L/2)^2 + y^2}} - \frac{+L/2}{\sqrt{(+L/2)^2 + y^2}} \right] \hat{\mathbf{k}}$$

$$= \frac{\mu_0 I}{2\pi y} \frac{L/2}{\sqrt{(L/2)^2 + y^2}} \hat{\mathbf{k}} = \frac{\mu_0 I}{2\pi y} \cos\theta\,\hat{\mathbf{k}},$$

(9.11.3)

in agreement with Eq. (9.1.6)

(ii) $L \gg x, y$ (infinite length limit): This gives back the expected infinite-length result:

$$\vec{\mathbf{B}} = -\frac{\mu_0 I}{4\pi y} \left[\frac{-L/2}{L/2} - \frac{+L/2}{L/2} \right] \hat{\mathbf{k}} = \frac{\mu_0 I}{2\pi y} \hat{\mathbf{k}}.$$ (9.11.4)

If we use cylindrical coordinates with the wire pointing along the $+z$-axis then the magnetic field is given by the expression

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r} \hat{\boldsymbol{\phi}}, \tag{9.11.5}$$

where $\hat{\boldsymbol{\phi}}$ is the tangential unit vector and the field point P is a distance r away from the wire.

9.11.2 Current-Carrying Arc

Consider the current-carrying loop formed of radial lines and segments of circles whose centers are at point P as shown below. Find the magnetic field $\vec{\mathbf{B}}$ at P.
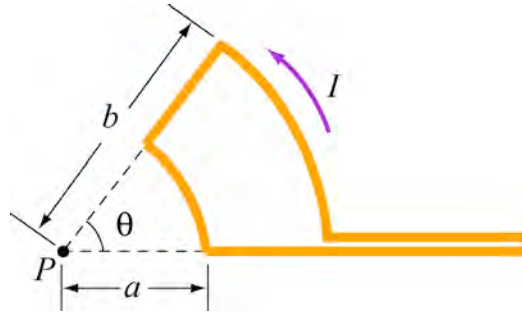
Figure 9.11.2 Current-carrying arc

Solution: According to the Biot-Savart law, the magnitude of the magnetic field due to a differential current-carrying element $I\,d\vec{\mathbf{s}}$ is given by

$$dB = \frac{\mu_0 I}{4\pi} \frac{|d\vec{\mathbf{s}} \times \hat{\mathbf{r}}|}{r^2} = \frac{\mu_0 I}{4\pi} \frac{r\,d\theta'}{r^2} = \frac{\mu_0 I}{4\pi r} d\theta'. \tag{9.11.6}$$

For the outer arc, we have

$$B_{\text{outer}} = \frac{\mu_0 I}{4\pi b} \int_0^\theta d\theta' = \frac{\mu_0 I \theta}{4\pi b}. \tag{9.11.7}$$

The direction of $\vec{\mathbf{B}}_{\text{outer}}$ is determined by the vector product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$, which points out of the plane of the figure. Similarly, for the inner arc, we have

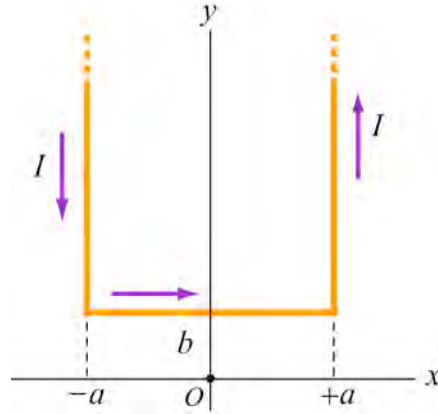$$B_{\text{inner}} = \frac{\mu_0 I}{4\pi a} \int_0^\theta d\theta' = \frac{\mu_0 I \theta}{4\pi a}. \tag{9.11.8}$$

For $\vec{\mathbf{B}}_{\text{inner}}$, $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ points into the plane of the figure. Thus, the total magnitude of magnetic field is

$$\vec{B} = \vec{B}_{inner} + \vec{B}_{outer} = \frac{\mu_0 I \theta}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{k} \ . \tag{9.11.9}$$

where $\hat{k}$ is a unit vector pointing into the plane of the figure.

9.11.3 Rectangular Current Loop

Determine the magnetic field (in terms of I, a and b) at the origin O due to the current loop shown in Figure 9.11.3.
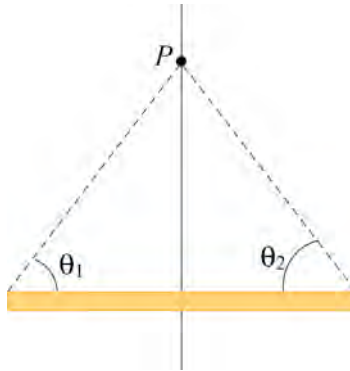

Figure 9.11.3 Rectangular current loop

Solution: For a finite wire carrying a current I, the contribution to the magnetic field at a point P is given by Eq. (9.1.5),

$$B = \frac{\mu_0 I}{4\pi r}(\cos\theta_1 + \cos\theta_2),$$

where θ_1 and θ_2 are the angles that parameterize the length of the wire.

To obtain the magnetic field at O, we make use of the above formula. The contributions can be divided into three parts:

(i) Consider the left segment of the wire that extends from $(x, y) = (-a, +\infty)$ to $(-a, +d)$. The angles which parameterize this segment give $\cos\theta_1 = 1$ (where $\theta_1 = 0$) and $\cos\theta_2 = -b/\sqrt{b^2 + a^2}$. Therefore,

$$B_1 = \frac{\mu_0 I}{4\pi a}(\cos\theta_1 + \cos\theta_2) = \frac{\mu_0 I}{4\pi a}\left(1 - \frac{b}{\sqrt{b^2 + a^2}}\right). \qquad (9.11.10)$$

The direction of $\vec{\mathbf{B}}_1$ is out of page, or $+\hat{\mathbf{k}}$.

(ii) Next, we consider the segment that extends from $(x, y) = (-a, +b)$ to $(+a, +b)$. Again, the (cosine of the) angles are given by

$$\cos\theta_2 = \cos\theta_1 = \frac{a}{\sqrt{a^2 + b^2}}. \qquad (9.11.11)$$

The magnetic field strength is then

$$B_2 = \frac{\mu_0 I}{4\pi b}\left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}}\right) = \frac{\mu_0 I a}{2\pi b\sqrt{a^2 + b^2}}. \qquad (9.11.12)$$

The direction of $\vec{\mathbf{B}}_2$ is into the page, or $-\hat{\mathbf{k}}$.

(iii) The third segment of the wire runs from $(x, y) = (+a, +b)$ to $(+a, +\infty)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 \qquad (9.11.13)$$

The direction of $\vec{\mathbf{B}}_3$ is again out of page, or $+\hat{\mathbf{k}}$.

The magnetic field is then the sum

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \frac{\mu_0 I}{2\pi a}\left(1 - \frac{b}{(a^2 + b^2)^{1/2}}\right)\hat{\mathbf{k}} - \frac{\mu_0 I a}{2\pi b(a^2 + b^2)^{1/2}}\hat{\mathbf{k}}$$
$$= \frac{\mu_0 I}{2\pi ab(a^2 + b^2)^{1/2}}(b(a^2 + b^2)^{1/2} - b^2 - a^2)\hat{\mathbf{k}}. \qquad (9.11.14)$$

In the limit $a \to 0$, the horizontal segment is absent, and the two semi-infinite wires carrying currents in the opposite direction overlap each other and their contributions completely cancel. Thus, the magnetic field vanishes in this limit.

9.11.4 Hairpin-Shaped Current-Carrying Wire

An infinitely long current-carrying wire is bent into a hairpin-like shape shown in Figure 9.11.4. Find the magnetic field at the point P that lies at the center of the half-circle.
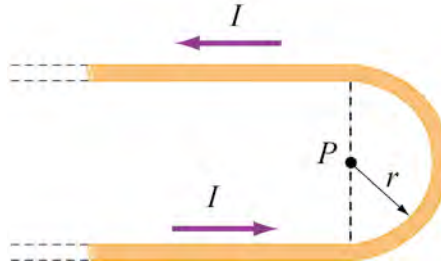
Figure 9.11.4 Hairpin-shaped current-carrying wire

Solution: Again we break the wire into three parts: two semi-infinite plus a semi-circular segments.

(i) Let P be located at the origin in the xy plane. The first semi-infinite segment then extends from $(x, y) = (-\infty, -r)$ to $(0, -r)$. The two angles which parameterize this segment are characterized by $\cos\theta_1 = 1$ (where $\theta_1 = 0$) and $\cos\theta_2 = 0$ (where $\theta_2 = \pi / 2$). Therefore, its contribution to the magnetic field at P is

$$B_1 = \frac{\mu_0 I}{4\pi r}(\cos\theta_1 + \cos\theta_2) = \frac{\mu_0 I}{4\pi r}(1 + 0) = \frac{\mu_0 I}{4\pi r}. \tag{9.11.15}$$

The direction of $\vec{\mathbf{B}}_1$ is out of page, or $+\hat{\mathbf{k}}$.

(ii) For the semi-circular arc of radius r, we make use of the Biot-Savart law:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}, \tag{9.11.16}$$

and obtain

$$B_2 = \frac{\mu_0 I}{4\pi} \int_0^\pi \frac{r d\theta}{r^2} = \frac{\mu_0 I}{4r}. \tag{9.11.17}$$

The direction of $\vec{\mathbf{B}}_2$ is out of page, or $+\hat{\mathbf{k}}$.

(iii) The third segment of the wire runs from $(x, y) = (0, +r)$ to $(-\infty, +r)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 = \frac{\mu_0 I}{4\pi r}. \tag{9.11.18}$$

The direction of $\vec{B}_3$ is again out of page, or $+\hat{k}$.
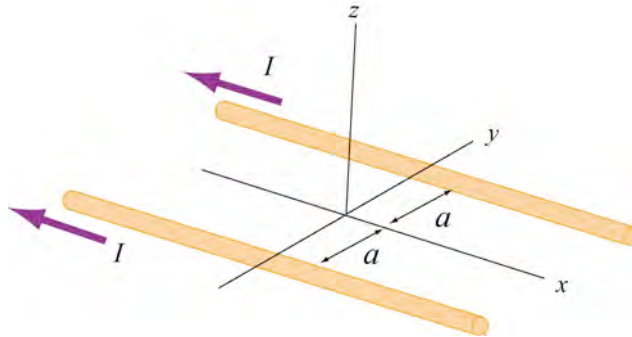
The total magnitude of the magnetic field is the sum

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 = 2\vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{2\pi r}\hat{k} + \frac{\mu_0 I}{4r}\hat{k} = \frac{\mu_0 I}{4\pi r}(2 + \pi)\hat{k}. \tag{9.11.19}$$

Notice that the contribution from the two semi-infinite wires is equal to that due to an infinite wire,

$$\vec{B}_1 + \vec{B}_3 = 2\vec{B}_1 = \frac{\mu_0 I}{2\pi r}\hat{k}. \tag{9.11.20}$$

9.11.5 Two Infinitely Long Wires

Consider two infinitely long wires carrying currents are in the negative x-direction.

Figure 9.11.5 Two infinitely long wires

(a) Plot the magnetic field pattern in the yz-plane.

(b) Find the distance d along the z-axis where the magnetic field is a maximum.

Solutions:

(a) The magnetic field lines are shown in Figure 9.11.6. Notice that the directions of both currents are into the page.
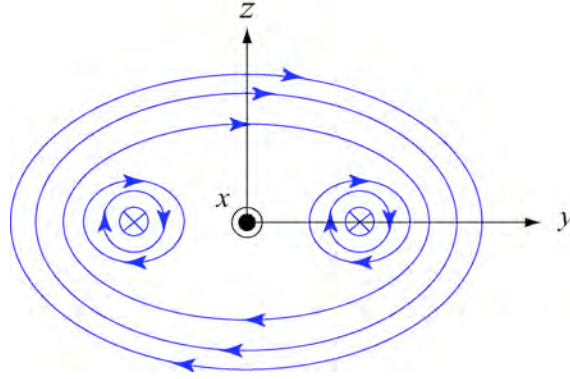
Figure 9.11.6 Magnetic field lines of two wires carrying current in the same direction.

(b) The magnetic field at $(0, 0, z)$ due to wire 1 on the left is, using Ampere's law,

$$B_1 = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi\sqrt{a^2 + z^2}}. \tag{9.11.21}$$

Because the current is in the negative x-direction, the magnetic field points in the direction of the vector product

$$-\hat{\mathbf{i}} \times \hat{\mathbf{r}}_1 = -\hat{\mathbf{i}} \times (\cos\theta\,\hat{\mathbf{j}} + \sin\theta\,\hat{\mathbf{k}}) = \sin\theta\,\hat{\mathbf{j}} - \cos\theta\,\hat{\mathbf{k}}. \tag{9.11.22}$$
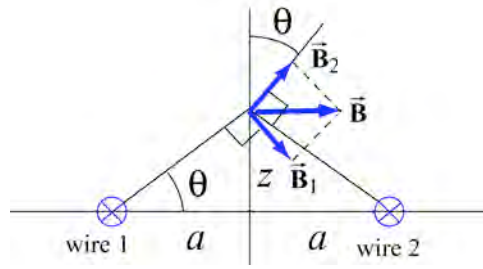
Thus, we have

$$\vec{\mathbf{B}}_1 = \frac{\mu_0 I}{2\pi\sqrt{a^2 + z^2}} (\sin\theta\,\hat{\mathbf{j}} - \cos\theta\,\hat{\mathbf{k}}). \tag{9.11.23}$$

For wire 2 on the right, the magnetic field strength is the same as the left one: $B_1 = B_2$. However, its direction is given by

$$-\hat{\mathbf{i}} \times \hat{\mathbf{r}}_2 = -\hat{\mathbf{i}} \times (-\cos\theta\,\hat{\mathbf{j}} + \sin\theta\,\hat{\mathbf{k}}) = \sin\theta\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}. \tag{9.11.24}$$

Adding up the contributions from both wires, the z-components cancel (as required by symmetry), and we arrive at

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \frac{\mu_0 I \sin\theta}{\pi\sqrt{a^2 + z^2}}\,\hat{\mathbf{j}} = \frac{\mu_0 I z}{\pi(a^2 + z^2)}\,\hat{\mathbf{j}}. \tag{9.11.25}$$

Figure 9.11.7 Superposition of magnetic fields due to two current sources.

To locate the maximum of B, we set $dB/dz = 0$ and find

$$\frac{dB}{dz} = \frac{\mu_0 I}{\pi}\left(\frac{1}{a^2 + z^2} - \frac{2z^2}{(a^2 + z^2)^2}\right) = \frac{\mu_0 I}{\pi}\frac{a^2 - z^2}{(a^2 + z^2)^2} = 0. \qquad (9.11.26)$$

The first derivative is zero when

$$z = a. \qquad (9.11.27)$$

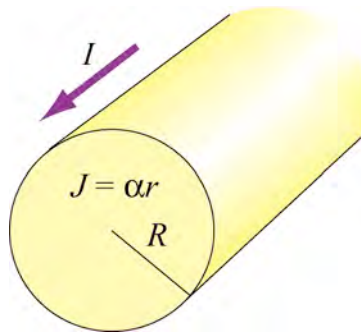Thus, at $z = a$, the magnetic field strength is a maximum, with a magnitude

$$B_{max} = \frac{\mu_0 I}{2\pi a}. \qquad (9.11.28)$$

9.11.6 Non-Uniform Current Density

Consider an infinitely long, cylindrical conductor of radius R carrying a current I with a non-uniform current density

$$J = \alpha r \qquad (9.11.29)$$

where α is a constant. Find the magnetic field everywhere.

Figure 9.11.8 Non-uniform current density.

Solution: The problem can be solved by using Ampere's law,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc},$$ (9.11.30)

where the enclosed current I_{enc} is given by

$$I_{enc} = \int \vec{J} \cdot d\vec{A} = \int (\alpha r')(2\pi r' dr').$$ (9.11.31)

(a) For $r < R$, the enclosed current is

$$I_{enc} = \int_0^r 2\pi \alpha r'^2 dr' = \frac{2\pi \alpha r^3}{3}.$$ (9.11.32)

Applying Ampere's law, the magnetic field at P_1 is given by

$$B_1(2\pi r) = \frac{2\mu_0 \pi \alpha r^3}{3}.$$ (9.11.33)

The magnetic field strength is

$$B_1 = \frac{\alpha \mu_0}{3} r^2, \qquad r < R.$$ (9.11.34)

The direction of the magnetic field $\vec{B}_1$ is tangential to the Amperian loop that encloses the current.

(b) For $r > R$, the enclosed current is

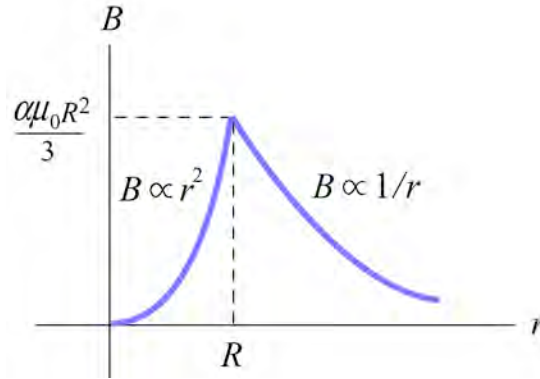$$I_{enc} = \int_0^R 2\pi \alpha r'^2 dr' = \frac{2\pi \alpha R^3}{3},$$ (9.11.35)

which yields

$$B_2(2\pi r) = \frac{2\mu_0 \pi \alpha R^3}{3}.$$ (9.11.36)

Thus, the magnetic field at a point P_2 outside the conductor is

$$B_2 = \frac{\alpha \mu_0 R^3}{3r}, \qquad r > R.$$ (9.11.37)

A plot of B as a function of r is shown in Figure 9.11.9.

Figure 9.11.9 The magnetic field as a function of distance away from the conductor.

9.11.7 Thin Strip of Metal

Consider an infinitely long, thin strip of metal of width w lying in the xy plane. The strip carries a current I along the $+x$-direction, as shown in Figure 9.11.10. Find the magnetic field at a point P that lies is in the plane of the strip and at a distance s away from it.
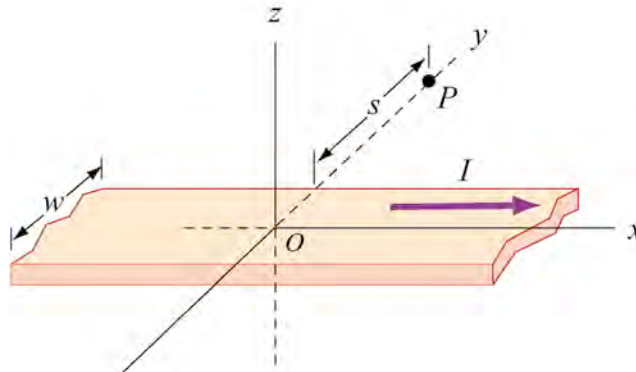
Figure 9.11.10 Thin strip of metal.

Solution: Consider a thin strip of width dr parallel to the direction of the current and at a distance r away from P, as shown in Figure 9.11.11. The amount of current carried by this differential element is

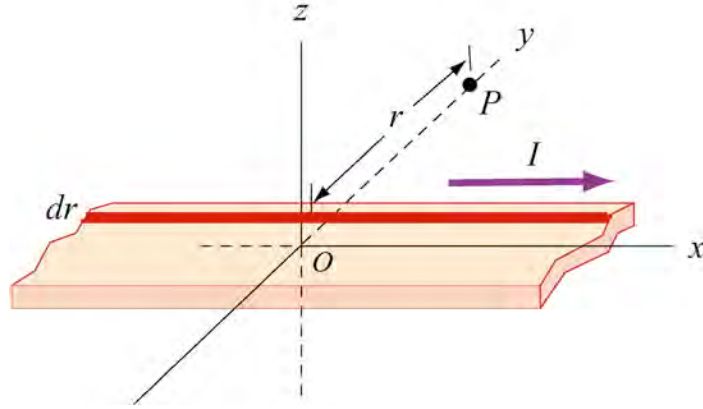$$dI = I \frac{dr}{w}.$$
(9.11.38)

Using Ampere's law, we see that the strip's contribution to the magnetic field at P is given by

$$dB(2\pi r) = \mu_0 I_{\text{enc}} = \mu_0 (dI).$$
(9.11.39)

The contribution to the magnetic field strength is

$$dB = \frac{\mu_0 \, dI}{2\pi r} = \frac{\mu_0}{2\pi r} \frac{I \, dr}{w}. \tag{9.11.40}$$

Figure 9.11.11 A thin strip with thickness dr carrying a steady current I.

Integrating this expression, we obtain

$$B = \int_s^{s+w} \frac{\mu_0 I}{2\pi w} \frac{dr}{r} = \frac{\mu_0 I}{2\pi w} \ln\left(\frac{s+w}{s}\right). \tag{9.11.41}$$

Using the right-hand rule, the direction of the magnetic field can be shown to point in the positive z-direction,

$$\vec{B} = \frac{\mu_0 I}{2\pi w} \ln\left(1 + \frac{w}{s}\right) \hat{k}. \tag{9.11.42}$$

In the limit of vanishing width, $w \ll s$, $\ln(1 + w/s) \approx w/s$, and the above expression becomes

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{k}, \tag{9.11.43}$$

corresponding to the magnetic field of an infinitely long thin straight wire.

9.11.8 Two Semi-Infinite Wires

A wire carrying current I directed down the y-axis to the origin, thence out to infinity along the positive x-axis. Show that the magnetic field in the quadrant with $x, y > 0$ of the xy-plane is given by

$$B_z = \frac{\mu_0 I}{4\pi} \left(\frac{1}{x} + \frac{1}{y} + \frac{x}{y\sqrt{x^2+y^2}} + \frac{y}{x\sqrt{x^2+y^2}} \right). \qquad (9.11.44)$$

Solution: Let $P(x, y)$ be a point in the first quadrant at a distance r_1 from a point $(0, y')$ on the y-axis and distance r_2 from $(x', 0)$ on the x-axis.
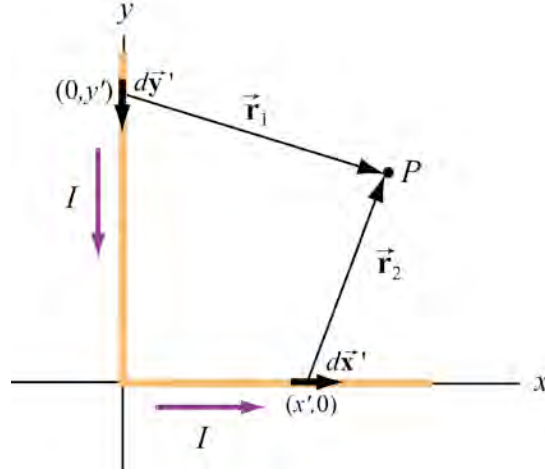
Figure 9.11.12 Two semi-infinite wires.

Using the Biot-Savart law, the magnetic field at P is given by

$$\vec{\mathbf{B}} = \int d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int_{\text{wire } y} \frac{d\vec{\mathbf{s}}_1 \times \hat{\mathbf{r}}_1}{r_1^2} + \frac{\mu_0 I}{4\pi} \int_{\text{wire } x} \frac{d\vec{\mathbf{s}}_2 \times \hat{\mathbf{r}}_2}{r_2^2}. \qquad (9.11.45)$$

Let's analyze each segment separately.

(i) Along the y-axis, consider a differential element $d\vec{\mathbf{s}}_1 = -dy'\hat{\mathbf{j}}$ that is located at a distance $\vec{\mathbf{r}}_1 = x\hat{\mathbf{i}} + (y - y')\hat{\mathbf{j}}$ from P. This yields

$$d\vec{\mathbf{s}}_1 \times \vec{\mathbf{r}}_1 = (-dy'\hat{\mathbf{j}}) \times [x\hat{\mathbf{i}} + (y - y')\hat{\mathbf{j}}] = x\,dy'\hat{\mathbf{k}}. \qquad (9.11.46)$$

(ii) Similarly, along the x-axis, we have $d\vec{\mathbf{s}}_2 = dx'\hat{\mathbf{i}}$ and $\vec{\mathbf{r}}_2 = (x - x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and so

$$d\vec{\mathbf{s}}_2 \times \vec{\mathbf{r}}_2 = y\,dx'\hat{\mathbf{k}}. \qquad (9.11.47)$$

Thus, we see that the magnetic field at P points in the positive z-direction. Using the above results and $r_1 = [x^2 + (y - y')^2]^{1/2}$ and $r_2 = [(x - x')^2 + y^2]^{1/2}$, we obtain

$$B_z = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{x\,dy'}{[x^2 + (y - y')^2]^{3/2}} + \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{y\,dx'}{[y^2 + (x - x')^2]^{3/2}}. \qquad (9.11.48)$$

The integrals can be readily evaluated using

$$\int_0^\infty \frac{b\,ds}{[b^2 + (a - s)^2]^{3/2}} = \frac{1}{b} + \frac{a}{b\sqrt{a^2 + b^2}}. \qquad (9.11.49)$$

The final expression for the magnetic field is given by

$$\vec{B} = \frac{\mu_0 I}{4\pi} \left[\frac{1}{x} + \frac{y}{x\sqrt{x^2 + y^2}} + \frac{1}{y} + \frac{x}{y\sqrt{x^2 + y^2}} \right] \hat{k}. \qquad (9.11.50)$$

A little trigonometry reveals that Eq. (9.11.50) is consistent with Eq. (9.1.5).

9.12 Conceptual Questions

1. Compare and contrast Biot-Savart law with Coulomb's law.

2. If a spring carries a current, does the spring stretch or compress? Explain.

3. How is the path of the integration of $\oint \vec{B} \cdot d\vec{s}$ chosen when applying Ampere's law?

4. Two concentric, co-planar circular loops of different diameters carry steady currents in the same direction. Do the loops attract or repel each other? Explain.

5. Suppose three infinitely long parallel wires are arranged in such a way that when looking at the cross section, they are at the corners of an equilateral triangle. Can currents be arranged (combination of flowing in or out of the page) so that all three wires (a) attract, and (b) repel each other? Explain.


9.13 Additional Problems

9.13.1 Application of Ampere's Law

The simplest possible application of Ampere's law allows us to calculate the magnetic field in the vicinity of a single infinitely long wire. Adding more wires with differing currents will check your understanding of Ampere's law.

(a) Calculate with Ampere's law the magnetic field, $|\vec{\mathbf{B}}| = B(r)$, as a function of distance r from the wire, in the vicinity of an infinitely long straight wire that carries current I. Show with a sketch the integration path you choose and state explicitly how you use symmetry. What is the field at a distance of 10 mm from the wire if the current is 10 A?

(b) Eight parallel wires cut the page perpendicularly at the points shown. A wire labeled with the integer k (k = 1, 2, ... , 8) bears the current $2k$ times I_0 (i.e., $I_k = 2k I_0$). For those with $k = 1$ to 4, the current flows up out of the page; for the rest, the current flows down into the page. Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ along the closed path (see figure) in the direction indicated by the arrowhead. (Watch your signs!)
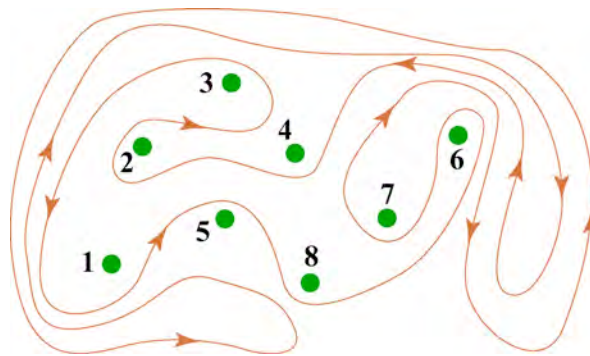

Figure 9.13.1 Amperian loop.

(c) Can you use a single application of Ampere's Law to find the field at a point in the vicinity of the 8 wires? Why? How would you proceed to find the field at an arbitrary point P?

9.13.2 Magnetic Field of a Current Distribution from Ampere's Law

Consider the cylindrical conductor with a hollow center and copper walls of thickness $b - a$ as shown in Figure 9.13.2. The radii of the inner and outer walls are a and b respectively, and the current I is uniformly spread over the cross section of the copper.

(a) Calculate the magnitude of the magnetic field in the region outside the conductor, $r > b$. (Hint: consider the entire conductor to be a single thin wire, construct an Amperian loop, and apply Ampere's Law.) What is the direction of $\vec{\mathbf{B}}$?
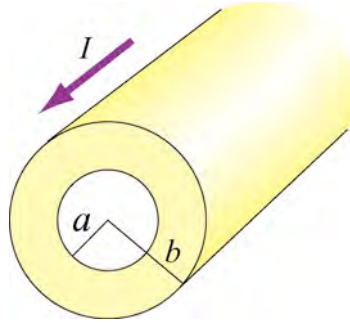
Figure 9.13.2 Hollow cylinder carrying a steady current I.

(b) Calculate the magnetic field inside the inner radius, $r < a$. What is the direction of $\vec{\mathbf{B}}$?

(c) Calculate the magnetic field within the inner conductor, $a < r < b$. What is the direction of $\vec{\mathbf{B}}$?

(d) Plot the behavior of the magnitude of the magnetic field $B(r)$ from $r = 0$ to $r = 4b$. Is $B(r)$ continuous at $r = a$ and $r = b$? What about its slope?

(e) Now suppose that a very thin wire running down the center of the conductor carries the same current I in the opposite direction. Can you plot, roughly, the variation of $B(r)$ without another detailed calculation? (Hint: remember that the vectors $d\vec{\mathbf{B}}$ from different current elements can be added to obtain the magnetic field.)

9.13.3 Cylinder with a Hole

A long copper rod of radius a has an off-center cylindrical hole through its entire length, as shown in Figure 9.13.3. The conductor carries a current I that is directed out of the page and is uniformly distributed throughout the cross section. Find the magnitude and direction of the magnetic field at the point P.
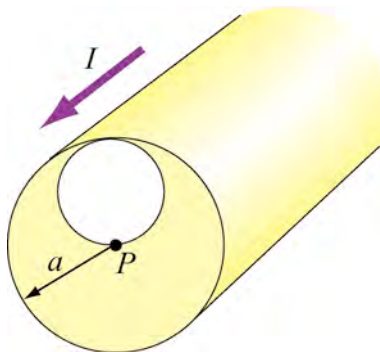
Figure 9.13.3 A cylindrical conductor with a hole.

9.13.4 The Magnetic Field Through a Solenoid

A solenoid has 200 closely spaced turns so that, for most of its length, it may be considered to be an ideal solenoid. It has a length of 0.25 m, a diameter of 0.1 m, and carries a current of 0.30 A.

(a) Sketch the solenoid, showing clearly the rotation direction of the windings, the current direction, and the magnetic field lines (inside and outside) with arrows to show their direction. What is the dominant direction of the magnetic field inside the solenoid?

(b) Find the magnitude of the magnetic field inside the solenoid by constructing an Amperian loop and applying Ampere's law.

(c) Does the magnetic field have a component in the direction of the wire in the loops making up the solenoid? If so, calculate its magnitude both inside and outside the solenoid, at radii 30 mm and 60 mm respectively, and show the directions on your sketch.

9.13.5 Rotating Disk

A circular disk of radius R with uniform charge density σ rotates with an angular speed ω. Show that the magnetic field at the center of the disk is

$$B = \frac{1}{2}\mu_0 \sigma \omega R .$$

Hint: Consider a circular ring of radius r and thickness dr. Show that the current in this element is $dI = (\omega/2\pi)dq = \omega\sigma r\, dr$.

9.13.6 Four Long Conducting Wires

Four infinitely long parallel wires carrying equal current I are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in Figure 9.13.5.
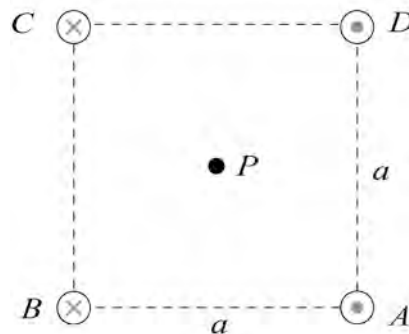

Figure 9.13.5 Four parallel conducting wires

Currents in A and D point out of the page, and into the page at B and C. What is the magnetic field at the center of the square?


9.13.7 Magnetic Force on a Current Loop

A rectangular loop of length l and width w carries a steady current I_1. The loop is then placed near a finitely long wire carrying a current I_2, as shown in Figure 9.13.6. What is the magnetic force experienced by the loop due to the magnetic field of the wire?
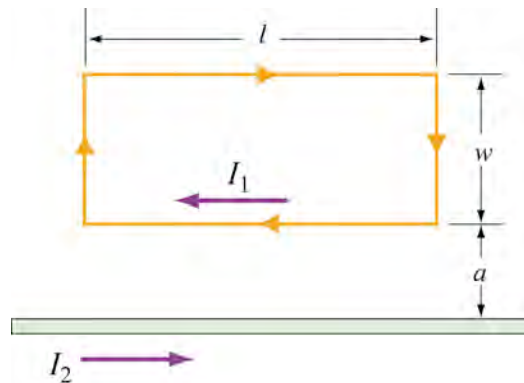
Figure 9.13.6 Magnetic force on a current loop.

9.13.8 Magnetic Moment of an Orbital Electron

We want to estimate the magnetic dipole moment associated with the motion of an electron as it orbits a proton. We use a "semi-classical" model to do this. Assume that the electron has speed v and orbits a proton (assumed to be very massive) located at the origin. The electron is moving in a right-handed sense with respect to the z-axis in a circle of radius $r = 0.53$ Å, as shown in Figure 9.13.7. Note that 1 Å $= 10^{-10}$ m .
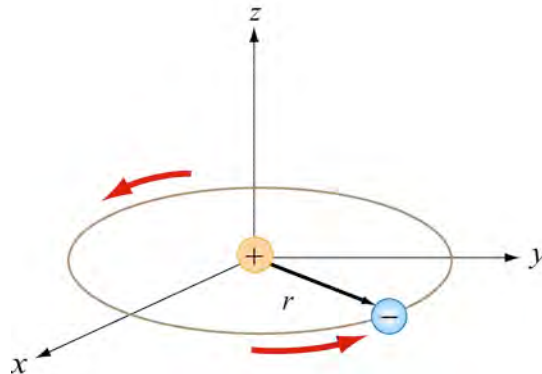
Figure 9.13.7

(a) The inward force necessary to required to make the electron move in this circle is provided by the Coulomb attractive force between the electron and proton (m_e is the mass of the electron). Using this fact, and the value of r we give above, find the speed of the electron in our "semi-classical" model. [Ans. 2.18×10^6 m/s.]

(b) Given this speed, what is the orbital period T of the electron? [Ans. 1.52×10^{-16}s .]

(c) What current is associated with this motion? Think of the electron as stretched out uniformly around the circumference of the circle. In a time T, the total amount of charge q that passes an observer at a point on the circle is just e . [Ans. 1.05 mA. Big!]

(d) What is the magnetic dipole moment associated with this orbital motion? Give the magnitude and direction. The magnitude of this dipole moment is called the *Bohr magneton,* μ_B. [Ans. 9.27×10^{-24} A$\cdot$m^2 along the negative z -axis.]

(e) One of the reasons this model is "semi-classical" is because classically there is no reason for the radius of the orbit above to assume the specific value we have given. The value of r is determined from quantum mechanical considerations. The orbital angular momentum of the electron can only assume integral multiples of $h/2\pi$, where $h = 6.63\times10^{-34}$ J$\cdot$s is the Planck constant. What is the orbital angular momentum of the electron in this model, in units of $h/2\pi$?

9.13.9 Ferromagnetism and Permanent Magnets

A disk of iron has a height $h = 1.00$ mm and a radius $r = 1.00$ cm . The magnetic dipole moment of an atom of iron is $\mu = 1.8\times10^{-23}$ A$\cdot$m^2. The molar mass of iron is 55.85 g, and its density is 7.9 g/cm^3. Assume that all the iron atoms in the disk have their dipole moments aligned with the axis of the disk.

(a) What is the number density of the iron atoms? How many atoms are in this disk? [Ans. 8.5×10^{28} atoms/m^3; 2.7×10^{22} atoms .]

(b) What is the magnetization $\vec{\mathbf{M}}$ in this disk? [Ans. 1.53×10^6 A/m , parallel to axis.]

(c) What is the magnetic dipole moment of the disk? [Ans. 0.48 A$\cdot$m^2 .]

(d) If we were to wrap one loop of wire around a circle of the same radius r, how much current would the wire have to carry to get the dipole moment in (c)? This is the "equivalent" surface current due to the atomic currents in the interior of the magnet. [Ans. 1525 A.]

9.13.10 Charge in a Magnetic Field

A coil of radius R with its symmetric axis along the x-direction carries a steady current I directed as shown in Figure 9.13.8. A positive charge q moves with a velocity $\vec{v} = v\,\hat{j}$ when it crosses the axis at a distance x from the center of the coil.
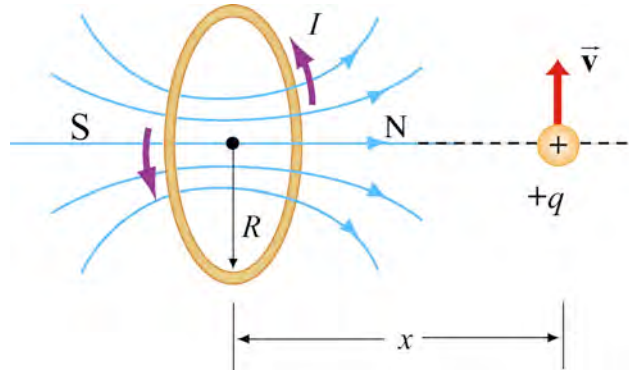

Figure 9.13.8

Describe the subsequent motion of the charge. What is the instantaneous radius of curvature?

9.13.11 Permanent Magnets

A magnet in the shape of a cylindrical rod has a length of 4.8 cm and a diameter of 1.1 cm. It has a uniform magnetization M of 5300 A/m, directed parallel to its axis.

(a) Calculate the magnetic dipole moment of this magnet.

(b) What is the axial field a distance of 1 meter from the center of this magnet, along its axis? [Ans. (a) 2.42×10^{-2} A·m^2, (b) 4.8×10^{-9} T, or 4.8×10^{-5} gauss .]

9.13.12 Magnetic Field of a Solenoid

(a) A 3000-turn solenoid has a length of 60 cm and a diameter of 8 cm. If this solenoid carries a current of 5.0 A, find the magnitude of the magnetic field inside the solenoid by constructing an Amperian loop and applying Ampere's Law. How does this compare to the magnetic field of the earth (0.5 gauss). [Ans. 0.0314 T, or 314 gauss, or about 600 times the magnetic field of the earth].

We make a magnetic field in the following way: we have a long cylindrical shell of non-conducting material which carries a surface charge fixed in place (glued down) with charge density σ, as shown in Figure 9.13.9. The cylinder is suspended in a manner such that it is rotating about its axis such that the speed of the surface of the cylinder is v_0.
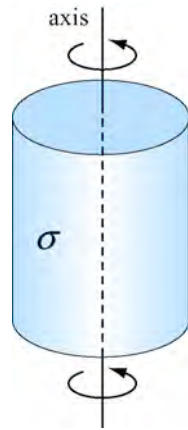
Figure 9.13.9

(b) What is the surface current K on the walls of the cylinder, in A/m? [Ans. $K = \sigma v_0$.]

(c) What is magnetic field inside the cylinder? [Ans. $B = \mu_0 K = \mu_0 \sigma v_0$, oriented along axis right-handed with respect to spin.]

(d) What is the magnetic field outside of the cylinder? Assume that the cylinder is infinitely long. [Ans. 0].

9.13.13 Effect of Paramagnetism

A solenoid with 16 turns/cm carries a current of 1.3 A.

(a) By how much does the magnetic field inside the solenoid increase when a close-fitting chromium rod is inserted? [Note: Chromium is a paramagnetic material with magnetic susceptibility $\chi = 2.7 \times 10^{-4}$.]

(b) Find the magnitude of the magnetization $\vec{\mathbf{M}}$ of the rod. [Ans. (a) 0.86 μT; (b) 0.68 A/m.]

9.14 Ampere's Law Simulation

In this section we explore the meaning of Ampere's Law using a 3D interactive simulation that creates imaginary, moveable Amperian loops in the presence of real, moveable line currents   This simulation illustrates Ampere's Law for a circular or rectangular imaginary Amperian loop, in the presence of current-carrying wires current currents both into and out of the page.
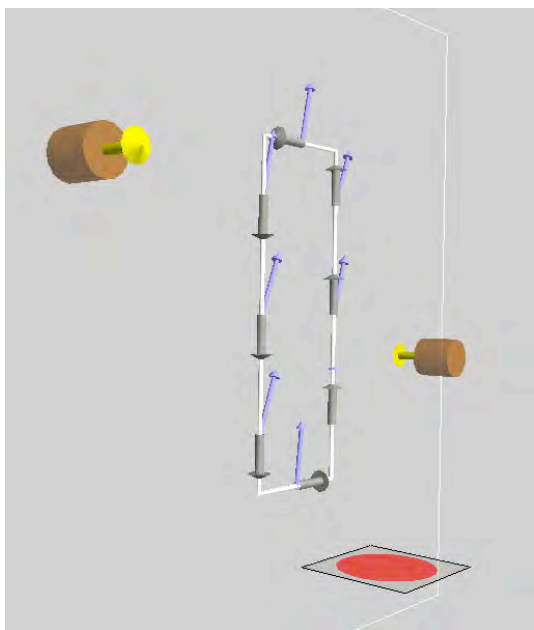
Figure 9.14.1 Screen Shot of Ampere's Law Simulation

You begin with one a wire carrying current out of the page and one carrying current into the page in the scene. You can add additional line currents, or delete all line currents present and start again. Left clicking and dragging on a line current can move that line current. You can choose whether your imaginary Amperian loop is a circle or a rectangle, and you can move that loop. You will see gray vectors $d\vec{s}$ tangent to the Amperian loops at many points on the loop. At those same points you will see the local magnetic field (blue vectors) on the loop due to all the line currents in the scene. If you left click and drag in the view, your perspective will change so that you can see the field vector and tangent orientation better. If you want to return to the original view you can "Reset Camera."

Use the simulation to verify the following properties of Ampere's Law. For the Amperian loop, you may choose either the circle or the rectangle.

(1) If line currents do not carry current through an Amperian loop, the line integral of the magnetic field around the loop is zero.

(2) If line currents do carry current through an Amperian loop, the line integral of the magnetic field around the loop is positive or negative depending on the direction of the total current penetrating the surface of the loop.

Then use the simulation to answer the two following questions. Consider two line currents. Place one of the charged line currents *inside* your Amperian loop and the other *outside*.

(1) Is the magnetic field **B** at any point on the loop due only to the line currents that

that are inside that loop?

(2) Is the dot product of the magnetic field **B** with the local line element $d\bar{\mathbf{s}}$ at any point on the loop due only to the magnetic fields associated with the line currents that are inside the loop?

(3) Is the *total* line integral of the magnetic field around the entire Amperean loop due only to the line currents that are inside the loop?