Chapter 7

DC Circuits

Chapter 7

Direct-Current Circuits

7.1 Introduction

Electrical circuits connect power supplies to *loads* such as resistors, capacitors, motors, heaters, or lamps. The connection between the supply and the load is made by soldering with wires that are often called *leads*, or with many kinds of connectors and terminals. Energy is delivered from the source to the user on demand at the flick of a switch. Sometimes many circuit elements are connected to the same lead, which is the called a *common lead* for those elements. Various parts of the circuits are called circuit elements, which can be in series or in parallel, as we have already seen in the case of capacitors.

A *node* is a point in a circuit where three or more elements are soldered together. A *branch* is a current path between two nodes. Each branch in a circuit can have only one current in it although a branch may have no current. A *loop* is a closed path that may consist of different branches with different currents in each branch.

A *direct current* (DC) circuit is a circuit is which the current through each branch in the circuit is always in the same direction. When the power supply is steady in time, and then the circuit is a purely resistive network then the current in each branch will be *steady*, that is the currents will not vary in time. In later chapters, when we introduce inductors into circuits with capacitance, transient power supplies initiate free *oscillating* currents. Finally when the power supply itself oscillates in time, then an *alternating current* (AC) is set up in the circuit.

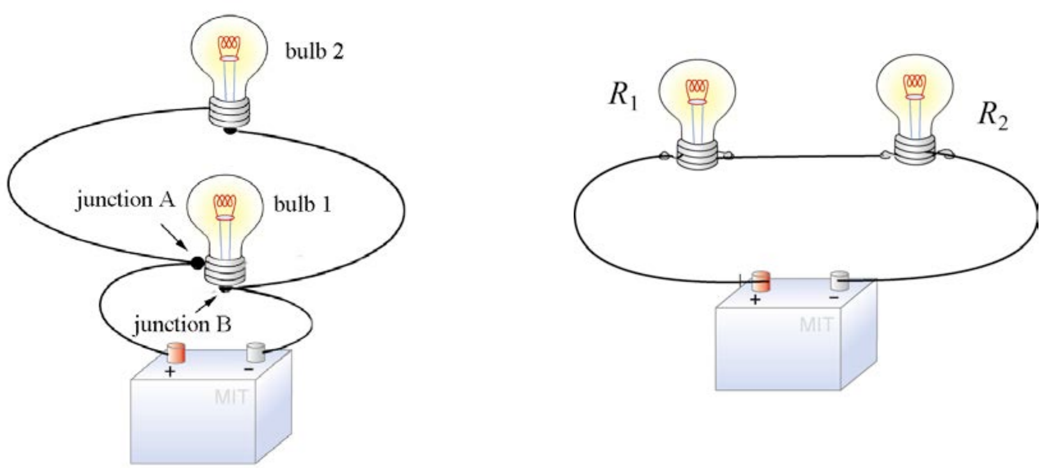


Figure 7.1.1 Elements connected (a) in parallel, and (b) in series.

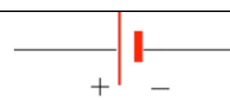
Example 7.1.1: Junctions, branches and loops

In the circuit shown in Figure 7.1.1(a), there are two junctions, A and B, on either side of light bulb 1. There are three branches: branch 1 goes from A to B through the battery, branch 2 goes from A to B through light bulb 1, and branch 3 goes from A to B through light bulb 2. There are three closed loops. We shall describe the loops by arbitrarily starting at junction A. Loop 1 consists of branches 1 and 2; it starts at junction A, passes through the battery to junction B, and then from junction B back to junction A through light bulb 1. Loop 2 consists of branches 2 and 3; it starts at junction A, passes through light bulb 1 to junction B, then continues through light bulb 2 back to junction A. Loop 3 consists of branches 1 and 3; it stars at junction A, passes through the battery to junction B, then continues through bulb 2 back to junction A. The circuit shown in Figure 7.1.1(b) has no junctions, one branch and one closed loop.

Elements are said to be in *parallel* when they are connected across the same potential difference. Both light bulbs in Figure 7.1.1(a) are connected across the battery. Generally, loads are connected in parallel across the power supply. On the other hand, when the elements are connected one after another in a branch, the same current passes through each element, and the elements are in *series* (see Figure 7.1.1b).

There are pictorial diagrams that show wires and components roughly as they appear, and schematic diagrams that use conventional symbols, somewhat like road maps. Some frequently used symbols are shown in the Table below.

Often there is a switch in series; when the switch is open the load is disconnected; when the switch is closed, the load is connected.

Electromotive Source Seat of emf	
Resistor	
Switch	

One can have closed circuits, through which current flows, or open circuits in which there are no currents. Usually by accident, wires may touch, causing a *short circuit*. Most of the current flows through the short, very little will flow through the load. This may burn out a piece of electrical equipment such as a transformer. To prevent damage, a fuse or circuit breaker is put in series. When there is a short the fuse blows, or the breaker opens.

In electrical circuits, a point (or some common lead) is chosen as the *ground*. This point is assigned an arbitrary voltage, usually zero, and the voltage V at any point in the circuit is defined as the potential difference between that point and ground.

7.2 Electromotive Force

Consider an electric circuit shown in Figure 7.2.1(a). To drive the current around the circuit, the battery undergoes a discharging process that converts chemical energy into electric energy that eventually gets dissipated as heat in the resistor.

In the external circuit, the electrostatic field, $\vec{\mathbf{E}}$, is directed from the positive terminal of the battery to the negative terminal of the battery, exerting a force on the charges in the wire to produce a current from the positive to the negative terminal. The electrostatic field also insures that the current in the wire is uniform. Recall that the electrostatic field is a conservative vector field and so the line integral around the loop in Figure 7.2.1(b) is zero,

$$\int_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0 \tag{7.2.1}$$

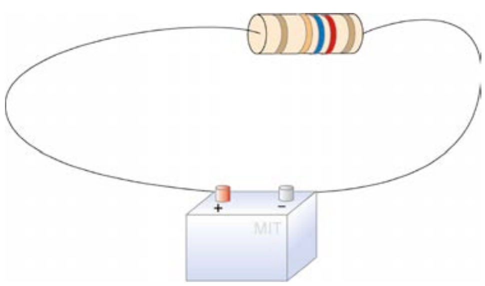


Figure 7.2.1(a) A simple circuit consisting of a battery and a resistor

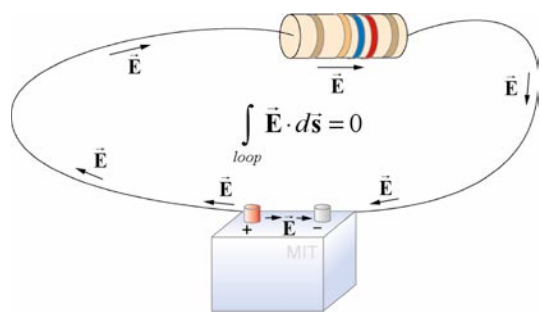


Figure 7.2.1(b) Integral of electrostatic field is zero around loop.

Inside the battery, in the region close to the positive terminal, the electrostatic field points away from the positive terminal. In the region close to the negative terminal, the electrostatic field points towards the negative terminal. (In the region in between, the electrostatic field may point in either direction depending on the nature of the battery.) The current is directed from the negative to the positive terminals. Near both terminals, the electrostatic field points in the opposite direction of the current.

In order to maintain the current, there must that be some force that transports charge carriers in the opposite direction in which the electrostatic field is trying to move them. The origin of this *source force*, $\vec{\mathbf{F}}_s$, in batteries is a chemical force. In the regions near the terminal where chemical reactions are taking place, chemical forces move charge carriers in the opposite direction in which the electrostatic field is trying to move them. The work done per unit charge by this source force in moving a charge carrier with charge q from the negative to the positive terminal is given by the expression,

$$\int_{neg}^{pos} \frac{\vec{\mathbf{F}}_s}{q} \cdot d\vec{\mathbf{s}} = \int_{neg}^{pos} \vec{\mathbf{f}}_s \cdot d\vec{\mathbf{s}}\,, \qquad (7.2.2)$$

where $\vec{\mathbf{f}}_s$ is the source force per unit charge. Outside the battery $\vec{\mathbf{f}}_s = \vec{\mathbf{0}}$, so we extend the path in Eq. (7.2.2) to the entire loop. In that case, the work done per unit charge by the non-electrostatic force, $\vec{\mathbf{F}}_s$, around a closed path is commonly referred to as the *electromotive force*, or *emf* (symbol ε).

$$\varepsilon \equiv \int_{\substack{closed\\path}} \vec{\mathbf{f}}_s \cdot d\vec{\mathbf{s}} = \int_{-}^{+} \frac{\vec{\mathbf{F}}_s}{q} \cdot d\vec{\mathbf{s}} = \int_{-}^{+} \vec{\mathbf{f}}_s \cdot d\vec{\mathbf{s}}\,. \qquad (7.2.3)$$

This is a poor choice of name because it is not a force but work done per unit charge. The SI unit for emf is the volt (V).

Inside our ideal battery without any internal resistance, the sum of the electrostatic force and the source force on the charge is zero,

$$q\vec{\mathbf{E}} + q\vec{\mathbf{f}}_s = \vec{\mathbf{0}}\,. \qquad (7.2.4)$$

Therefore the electrostatic field is equal in magnitude to the source force per unit charge but opposite in direction,

$$\vec{\mathbf{E}} = -\vec{\mathbf{f}}_s\,. \qquad (7.2.5)$$

The electric potential difference between the terminals is defined in terms of the electrostatic field

$$V(+) - V(-) = -\int_{-}^{+} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = \int_{-}^{+} \vec{\mathbf{f}}_s \cdot d\vec{\mathbf{s}} = \varepsilon\,. \qquad (7.2.6)$$

The potential difference ΔV between the positive and the negative terminals of the battery is called the *terminal voltage*, and in this case is equal to the emf .

Electromotive force is not restricted to chemical forces. In Figure 7.2.2, the inner working of a Van de Graaff generator are displayed. An electric motor drives a non-conducting belt that transports charge carriers in a direction opposite the electric field. The positive charge carriers are moved from lower to higher potential, and negative charge carriers are moved from higher to lower potential. Strong local fields at the brushes of the terminals both add and remove charge carriers from the belt. An electric motor provides the energy to move the belt and hence is the source of the electromotive force.

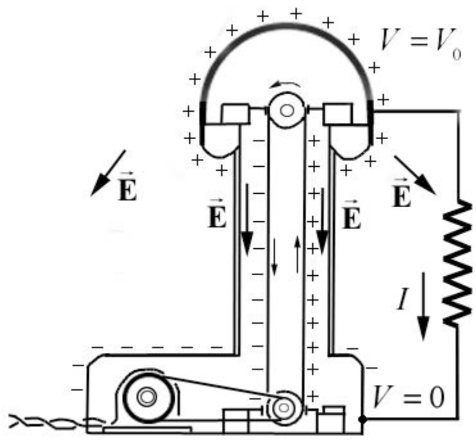


Figure 7.2.2 Van de Graaff generator

Solar cells and thermocouples are also examples of emf source. They can also be thought of as a “charge pump” that moves charges from lower potential to higher potential.

Consider a simple circuit consisting of a battery as the emf source and a resistor of resistance R, as shown in Figure 7.2.3.

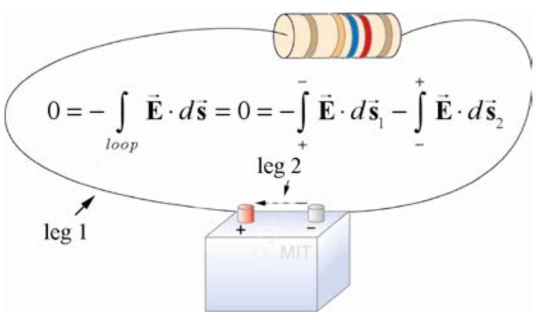


Figure 7.2.3(a) Electric potential difference for leg 1 and leg 2 sum to zero.

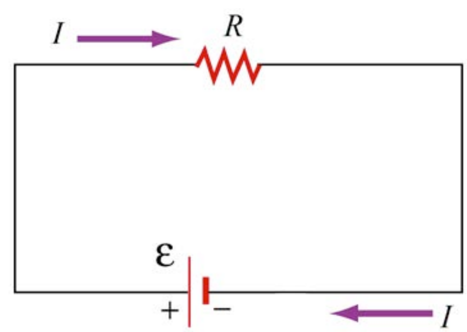


Figure 7.2.3(b) Circuit diagram.

The circuit diagram in Figure 7.2.3(b) corresponds to the circuit in Figure 7.2.3(a). The electric potential difference around the loop is zero because the electrostatic field is conservative, Eq. (7.2.1). We can divide the loop into two legs; leg 1 goes from the positive terminal to the negative terminal through the external circuit, and leg 2 goes from the negative terminal to the positive terminal through the battery,

$$0 = -\oint_{loop} \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = 0 = -\int_{+}^{-} \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}_1 - \int_{-}^{+} \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}_2 , \qquad (7.2.7)$$

The integral via leg 1 in the external circuit is just the potential difference across the resistor, which is given by Ohm’s Law, where we have assumed that the wires have negligible resistance,

$$\Delta V_1 = -\int_{+}^{-} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}_1 = -IR \,. \tag{7.2.8}$$

The integral via leg 2 through the battery is the emf (Eq. (7.2.6),

$$\Delta V_2 = -\int_{-}^{+} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = \varepsilon \,. \tag{7.2.9}$$

Eq. (7.2.7) becomes

$$0 = \Delta V_1 + \Delta V_2 = -IR + \varepsilon \,. \tag{7.2.10}$$

Therefore the current in the loop is given by

$$I = \frac{\varepsilon}{R} \,. \tag{7.2.11}$$

However, a real battery always carries an internal resistance r (Figure 7.2.4a), and the potential difference across the battery terminals becomes

$$\Delta V = \varepsilon - Ir \,. \tag{7.2.12}$$

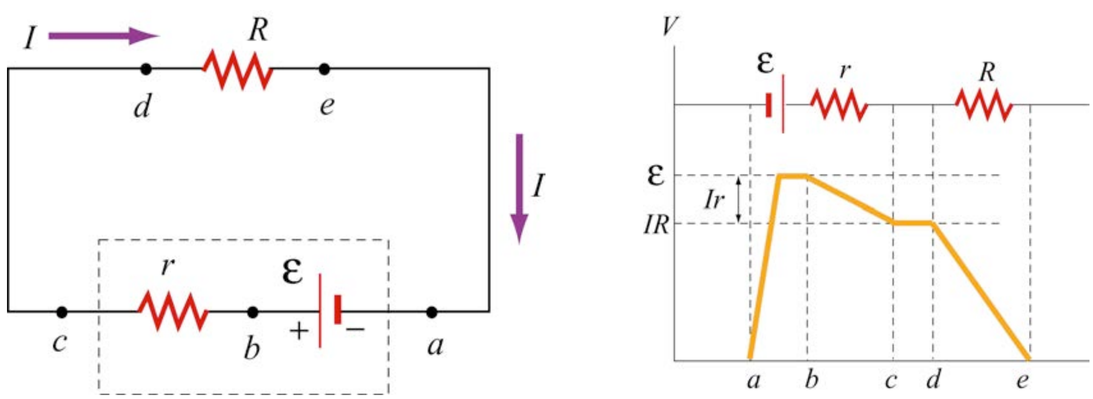


Figure 7.2.4 (a) Circuit with an emf source having an internal resistance r and a resistor of resistance R. (b) Change in electric potential around the circuit.

Because there is no net change in potential difference around a closed loop, we have

$$\varepsilon - Ir - IR = 0 \,. \tag{7.2.13}$$

Therefore the current through the circuit is

$$I = \frac{\varepsilon}{R + r}. \quad (7.2.14)$$

Figure 7.2.4(b) depicts the change in electric potential as we traverse the circuit clockwise. From the figure, we see that the highest potential is immediately after the battery. The potential drops as each resistor is crossed. Note that the potential is essentially constant along the wires. This is because the wires have a negligibly small resistance compared to the resistors.

7.3 Electrical Energy and Power

Consider a circuit consisting of an ideal battery (zero internal resistance) and a resistor with resistance *R* (Figure 7.3.1). The potential difference between two points *a* and *b* be $\varepsilon = V_b - V_a > 0$. If a charge Δq is moved through the battery, its electric potential energy is increased by $\Delta U = \Delta q \varepsilon$. On the other hand, as the charge moves across the resistor, the potential energy is decreased due to collisions with atoms in the resistor. If we neglect the internal resistance of the battery and the connecting wires, upon returning to *a,* the change in potential energy of Δq is zero.

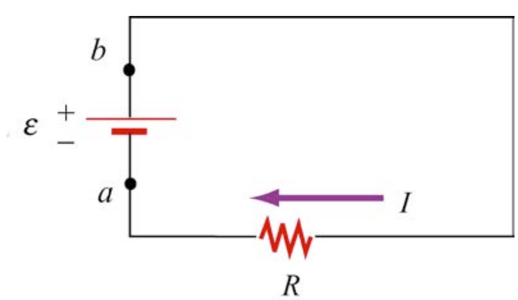


Figure 7.3.1 A circuit consisting of an ideal battery with emf ε and a resistor of resistance *R*.

The rate of energy loss through the resistor is given by

$$P = \frac{\Delta U}{\Delta t} = \left(\frac{\Delta q}{\Delta t}\right)\varepsilon = I\varepsilon. \quad (7.3.1)$$

This is equal to the power supplied by the battery. Using $\varepsilon = IR$ in Eq. (7.3.1), one may rewrite the rate of energy loss through the resistor as

$$P = I^2 R. \quad (7.3.2)$$

Using $I = \varepsilon / R$ in Eq. (7.3.1), the power delivered by the battery is

$$P = \varepsilon^2 / R. \quad (7.3.3)$$

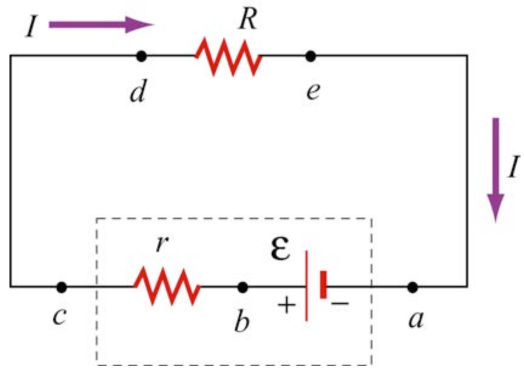


Figure 7.3.2 A circuit consisting of a battery with emf ε and a resistor of resistance *R*.

For a battery with emf ε and internal resistance r (Figure 7.3.2), the power or the rate at which chemical energy is delivered to the circuit is

$$P = I\varepsilon = I(IR + Ir) = I^2R + I^2r\,. \qquad (7.3.4)$$

The power of the source emf is equal to the sum of the power dissipated in both the internal and load resistance as required by energy conservation.

7.4 Resistors in Series and in Parallel

The two resistors with resistance R_1 and R_2 in Figure 7.4.1 are connected in series to a source of emf ε. By current conservation, the same current, I, is in each resistor.

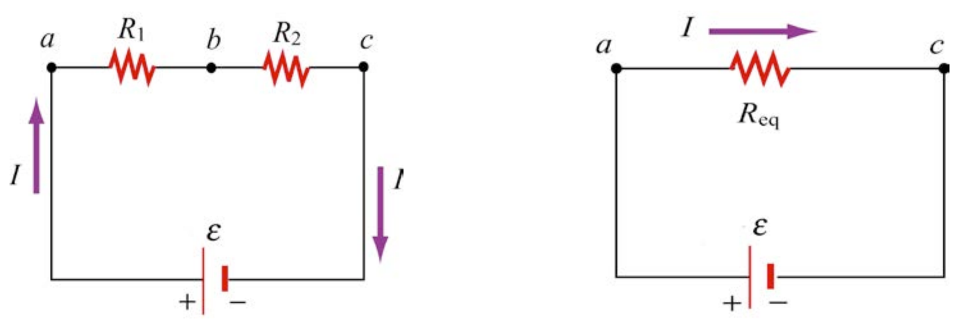


Figure 7.4.1 (a) Resistors in series. (b) Equivalent circuit.

The total voltage drop from *a* to *c* across both elements is the sum of the voltage drops across the individual resistors:

$$\varepsilon = \Delta V = I\,R_1 + I\,R_2 = I(R_1 + R_2)\,. \qquad (7.4.1)$$

The two resistors in series can be replaced by one equivalent resistor R_{eq} (Figure 7.4.1b) with the identical voltage drop $\varepsilon = \Delta V = I\,R_{\text{eq}}$ that implies that

$$R_{\text{eq}} = R_1 + R_2 \,. \tag{7.4.2}$$

The above argument can be extended to N resistors placed in series. The equivalent resistance is just the sum of the original resistances,

$$R_{\text{eq}} = R_1 + R_2 + \cdots = \sum_{i=1}^{N} R_i \,. \tag{7.4.3}$$

Notice that if one resistance R_1 is much larger than the other resistances R_i, then the equivalent resistance R_{eq} is approximately equal to the largest resistor R_1.

Next let's consider two resistors R_1 and R_2 that are connected in parallel across a source of emf ε, (Figure 7.4.2a).

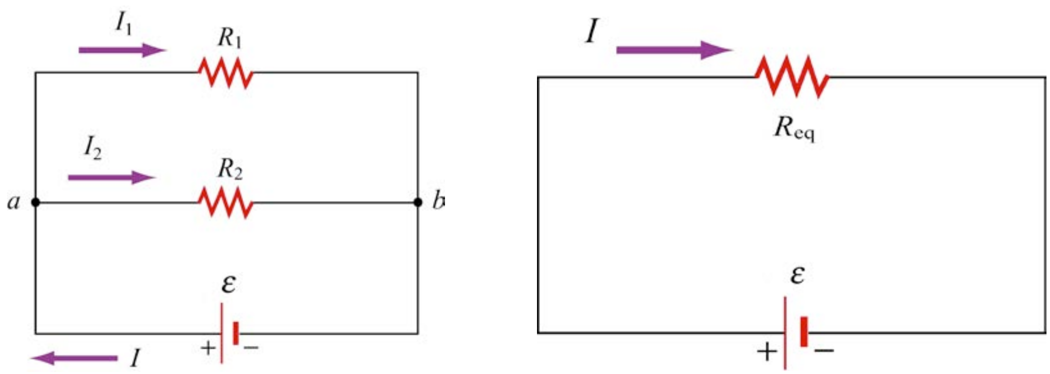


Figure 7.4.2 (a) Two resistors in parallel. (b) Equivalent resistance

By current conservation, the current I that passes through the source of emf must divide into a current I_1 that passes through resistor R_1 and a current I_2 that passes through resistor R_2. Each resistor individually satisfies Ohm's law, $\Delta V_1 = I_1\,R_1$ and $\Delta V_2 = I_2\,R_2$. However, the potential across the resistors are the same, $\Delta V_1 = \Delta V_2 = \varepsilon$. Current conservation then implies

$$I = I_1 + I_2 = \frac{\varepsilon}{R_1} + \frac{\varepsilon}{R_2} = \varepsilon\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{7.4.4}$$

The two resistors in parallel can be replaced by one equivalent resistor R_{eq} with $\varepsilon = IR_{\text{eq}}$ (Figure 7.4.2b). Comparing these results, the equivalent resistance for two resistors that are connected in parallel is given by

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \quad . \tag{7.4.5}$$

This result easily generalizes to N resistors connected in parallel

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{i=1}^{N} \frac{1}{R_i} . \tag{7.4.6}$$

When one resistance R_1 is much smaller than the other resistances R_i, then the equivalent resistance R_{eq} is approximately equal to the smallest resistor R_1. In the case of two resistors,

$$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2} \approx \frac{R_1 R_2}{R_2} = R_1 .$$

This means that almost all of the current that enters the node point will pass through the branch containing the smallest resistance. So, when a short develops across a circuit, all of the current passes through this path of nearly zero resistance.

7.5 Kirchhoff's Circuit Rules

In analyzing circuits, there are two fundamental (Kirchhoff's) rules.

Junction Rule

At any point where there is a node formed by the junction of various current carrying branches, by current conservation, the sum of the currents into the node must equal the sum of the currents out of the node (otherwise charge would build up at the junction);

$$\sum I_{\text{in}} = \sum I_{\text{out}} \ . \tag{7.5.1}$$

As an example, consider Figure 7.5.1 below:

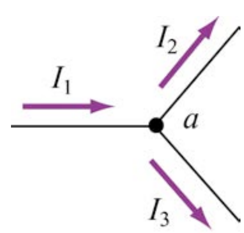


Figure 7.5.1 Kirchhoff's junction rule.

According to the junction rule, the three currents are related by

$$I_1 = I_2 + I_3 .$$

Loop Rule

The sum of the voltage drops ΔV, across any circuit elements that form a closed circuit is zero:

$$\sum_{\text{closed loop}} \Delta V = 0 . \tag{7.5.2}$$

The rules for determining ΔV across a resistor and a battery with a designated travel direction are shown below:

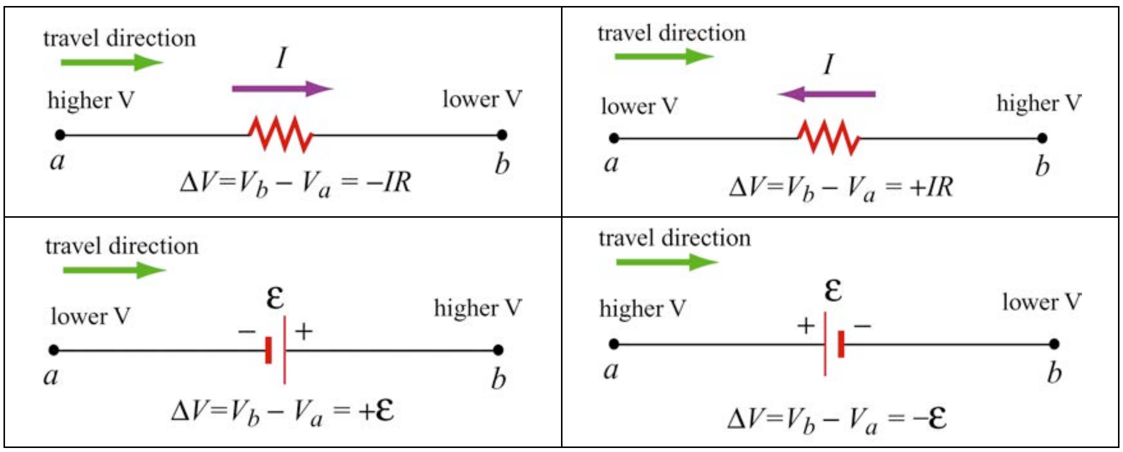


Figure 7.5.2 Rules for determining potential difference across resistors and batteries.

Note that the choice of travel direction is arbitrary. The same equation is obtained whether the closed loop is traversed clockwise or counterclockwise.

Example 7.5.1: Voltage divider

Consider a source of emf $\varepsilon = V_{in}$ that is connected in series to two resistors, R_1 and R_2

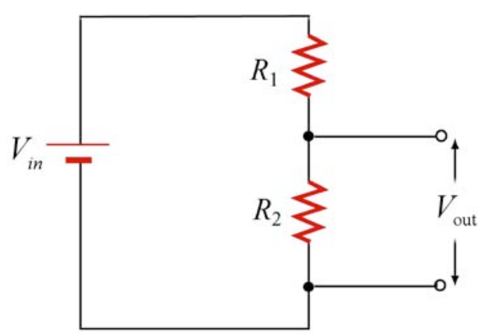


Figure 7.5.3 Voltage divider.

The potential difference, V_{out}, across resistor R_2 will be less than V_{in}. This circuit is called a *voltage divider*. From the loop rule,

$$V_{in} - IR_1 - IR_2 = 0 . \tag{7.5.3}$$

Therefore the current in the circuit is given by

$$I = \frac{V_{in}}{R_1 + R_2} \tag{7.5.4}$$

Thus the potential difference, V_{out}, across resistor R_2 is given by

$$V_{out} = IR_2 = \frac{R_2}{R_1 + R_2} V_{in} . \tag{7.5.5}$$

Note that the ratio of the potential differences characterizes the voltage divider and is determined by the resistors:

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2} \tag{7.5.6}$$

7.6 Voltage-Current Measurements

Any instrument that measures potential difference or current will disturb the circuit under observation. In some devices, known as ammeters, the current in a coil will cause meter movement (arising from the torque on a magnetic dipole in an magnetic field, a topic will soon study) or some change will result in a digital display. There will be some potential difference due to the resistance of the current through the ammeter. An ideal ammeter has zero resistance. However in the case of an ammeter that has resistance of 1Ω on the 250 mA range. The drop of 0.25 V may or may not be negligible; knowing the meter resistance allows one to correct for its effect on the circuit.

An ammeter can be converted to a voltmeter by putting a resistor R in series with the coil movement. The potential difference across some circuit element can be determined by connecting the coil movement and resistor in parallel with the circuit element. This causes a small amount of current to flow through the coil movement. The voltage drop across the element can now be determined by measuring I and computing the voltage drop from $\Delta V = IR$, which is read on a calibrated scale. The larger the resistance R, the smaller the amount of current is diverted through the coil. Thus an ideal voltmeter would have an infinite resistance.

Resistor Value Chart					
0	Black	4	Yellow	8	Gray
1	Brown	5	Green	9	White
2	Red	6	Blue	−1	Gold
3	Orange	7	Violet	−2	Silver

The colored bands on a composition resistor specify numbers according to the chart above (2-7 follow the rainbow spectrum). Starting from the end to which the bands are closest, the first two numbers specify the significant figures of the value of the resistor and the third number represents a power of ten by which the first two numbers are to be multiplied (gold is 10^{-1}). The fourth specifies the "tolerance," or precision, gold being 5% and silver 10%. As an example, a 43-Ω (43 ohms) resistor with 5% tolerance is represented by yellow, orange, black, and gold.

7.7 Capacitors in Electric Circuits

A capacitor can be charged by connecting the plates to the terminals of a battery, which are maintained at a potential difference ΔV, (terminal voltage).

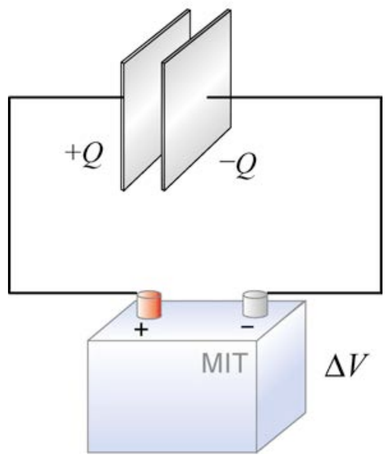


Figure 7.7.1 Charging a capacitor.

The connection results in sharing the charges between the terminals and the plates. For example, the plate that is connected to the positive terminal will acquire some positive charge, the plate that is connected to the negative terminal will acquire some negative charge. The sharing causes a momentary reduction of charges on the terminals, and a decrease in the terminal voltage. Chemical reactions are then triggered to transfer more charge from one terminal to the other to compensate for the loss of charge to the capacitor plates, and maintain the terminal voltage at its initial level. The battery can thus be thought of as a charge pump that brings a charge Q from one plate to the other.

7.7.1 Parallel Connection

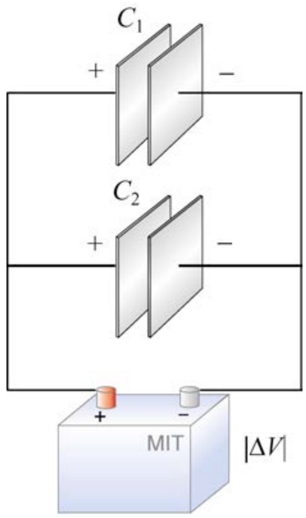


Figure 7.7.2(a) Capacitors connected in parallel.

Suppose we have two capacitors C_1 with charge Q_1 and C_2 with charge Q_2 that are connected in parallel, as shown in Figure 7.7.2(a). The left plates of both capacitors C_1 and C_2 are connected to the positive terminal of the battery and have the same electric potential as the positive terminal. Similarly, both right plates are negatively charged and have the same potential as the negative terminal. Thus, the potential difference $|\Delta V|$ is the same across each capacitor. This gives

$$C_1 = \frac{Q_1}{|\Delta V|}, \quad C_2 = \frac{Q_2}{|\Delta V|}. \tag{7.7.1}$$

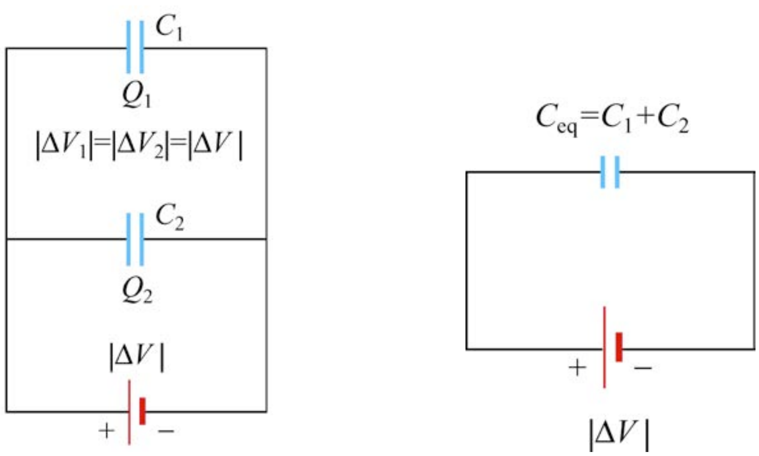


Figure 7.7.3(b) Capacitors in parallel and an equivalent capacitor.

These two capacitors can be replaced by a single equivalent capacitor C_{eq} with a total charge Q supplied by the battery (Figure 7.7.2(b)). However, since Q is shared by the two capacitors, we must have

$$Q = Q_1 + Q_2 = C_1 \,|\Delta V| + C_2 \,|\Delta V| = \left(C_1 + C_2\right)|\Delta V| . \qquad (7.7.2)$$

The equivalent capacitance is given by

$$C_{\text{eq}} = \frac{Q}{|\Delta V|} = C_1 + C_2 . \qquad (7.7.3)$$

Thus, capacitors that are connected in parallel add. The generalization to any number of capacitors is

$$\boxed{C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots + C_N = \sum_{i=1}^{N} C_i \qquad \text{(parallel)}} . \qquad (7.7.4)$$

7.7.2 Series Connection

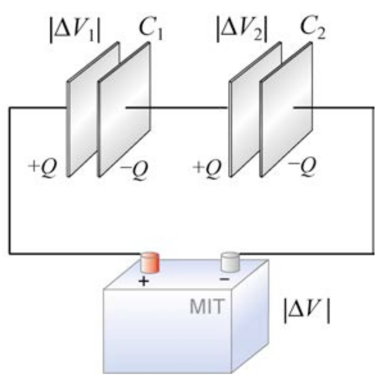


Figure 7.7.3(a) Capacitors in series and an equivalent capacitor.

Suppose two initially uncharged capacitors C_1 and C_2 are connected in series, as shown in Figure 7.7.3(a). A potential difference $|\Delta V|$ is then applied across both capacitors. The left plate of capacitor 1 is connected to the positive terminal of the battery and becomes positively charged with a charge $+Q$, while the right plate of capacitor 2 is connected to the negative terminal and becomes negatively charged with charge $-Q$ as electrons flow in. What about the inner plates? They were initially uncharged; now the outside plates each attract an equal and opposite charge. So the right plate of capacitor 1 will acquire a charge $-Q$ and the left plate of capacitor $+Q$.

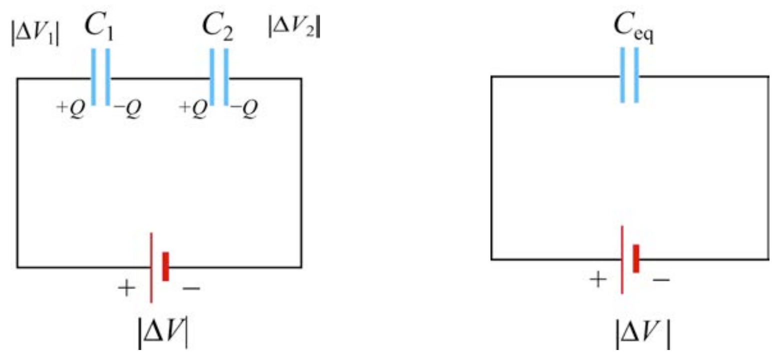


Figure 7.7.3(b) Capacitors in series and an equivalent capacitor.

The potential differences across capacitors C_1 and C_2 are

$$|\Delta V_1| = \frac{Q}{C_1}, \quad |\Delta V_2| = \frac{Q}{C_2}. \tag{7.7.5}$$

respectively. From Figure 7.7.3(b), the total potential difference is the sum of the two individual potential differences:

$$|\Delta V| = |\Delta V_1| + |\Delta V_2|. \tag{7.7.6}$$

In fact, the potential difference across any number of capacitors in series connection is equal to the sum of potential differences across the individual capacitors. These two capacitors can be replaced by a single equivalent capacitor $C_{\text{eq}} = Q / |\Delta V|$. Using the fact that the potentials add in series,

$$\frac{Q}{C_{\text{eq}}} = \frac{Q}{C_1} + \frac{Q}{C_2}.$$

and so the equivalent capacitance for two capacitors in series becomes

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{7.7.7}$$

The generalization to any number of capacitors connected in series is

$$\boxed{\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i} \qquad (\text{series})}. \tag{7.7.8}$$

Example 7.7.1: Equivalent Capacitance

Find the equivalent capacitance for the combination of capacitors shown in Figure 7.7.4(a)

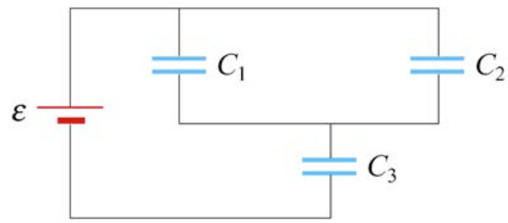
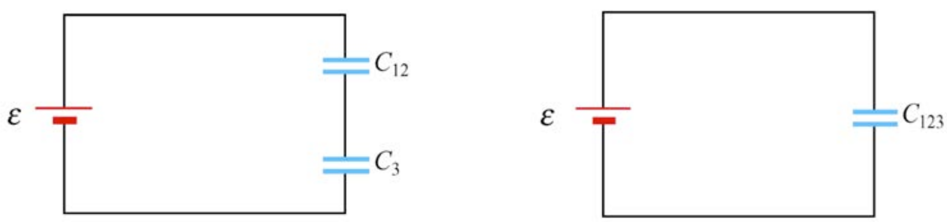


Figure 7.7.4 (a) Capacitors connected in series and in parallel

Solution: Because C_1 and C_2 are connected in parallel, their equivalent capacitance C_{12} is given by

$$C_{12} = C_1 + C_2 .$$



ε **Figure 7.7.4** (b) and (c) Equivalent circuits.

Now capacitor C_{12} is in series with C_3, as seen from Figure 7.7.4(b). So, the equivalent capacitance C_{123} is given by

$$\frac{1}{C_{123}} = \frac{1}{C_{12}} + \frac{1}{C_3} ,$$

or

$$\boxed{C_{123} = \frac{C_{12}C_3}{C_{12} + C_3} = \frac{(C_1 + C_2)C_3}{C_1 + C_2 + C_3}}.$$

We also comment that the configuration is equivalent to two capacitors connected in series, as shown in Figure 7.7.4(a).

Example 7.7.2: Capacitance with Dielectrics

A non-conducting slab of thickness t, area A and dielectric constant κ_e is inserted into the space between the plates of a parallel-plate capacitor with spacing d, charge Q and area A, as shown in Figure 7.7.5(a). The slab is not necessarily halfway between the capacitor plates. What is the capacitance of the system?

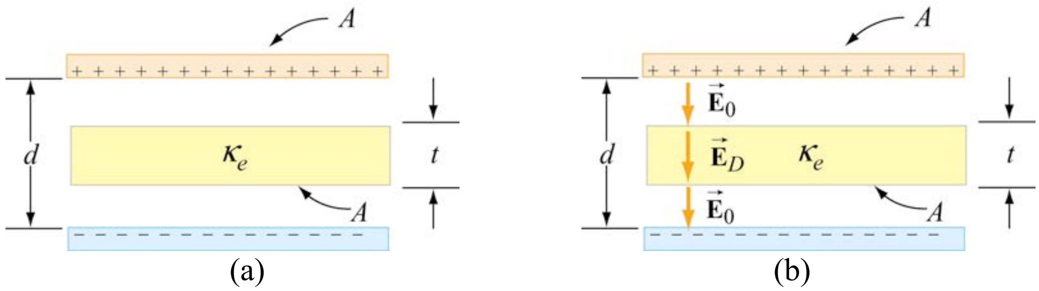


Figure 7.7.5 (a) Capacitor with a dielectric. (b) Electric field between the plates.

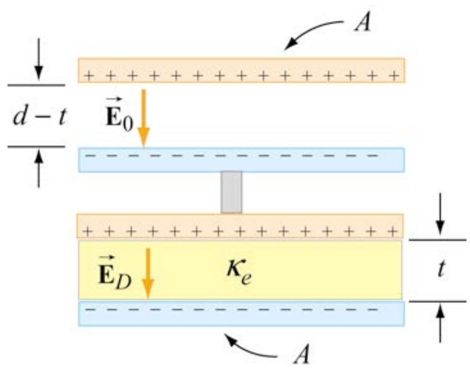


Figure 7.7.6 Equivalent configuration.

Using Eq. ((7.7.7)) for capacitors connected in series and Eq. 5.4.19 for the capacitance of a parallel plate capacitor with a dielectric, we have that

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d-t}{\varepsilon_0 A} + \frac{t}{\kappa_e \varepsilon_0 A}. \tag{7.7.9}$$

Therefore the equivalent capacitance is

$$C_{eq} = \frac{(\varepsilon_0 A)(\kappa_e \varepsilon_0 A)}{(d-t)\kappa_e \varepsilon_0 A + t\varepsilon_0 A}. \tag{7.7.10}$$

7.8 RC Circuit

We now consider capacitive circuits with a steady source of emf in which the current will vary in time.

7.8.1 Charging a Capacitor

Consider the circuit shown below. The capacitor is connected to a steady source of emf ε that does not vary in time. At time $t = 0$, the switch S is closed. The capacitor initially is uncharged, $q(t=0) = 0$. In particular for $t < 0$, there is no potential difference across the capacitor.

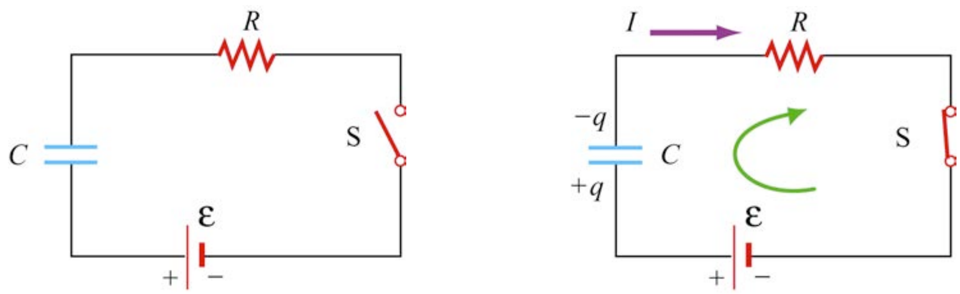


Figure 7.8.1 (a) RC circuit diagram for $t < 0$. (b) Circuit diagram for $t > 0$.

At $t = 0$, the switch is closed and current begins to flow according to

$$I_0 = \frac{\varepsilon}{R}. \qquad (7.8.1).$$

At this instant, the potential difference from the battery terminals is the same as that across the resistor. This initiates the charging of the capacitor. As the capacitor starts to charge, the potential difference across the capacitor increases in time

$$V_C(t) = \frac{q(t)}{C}. \qquad (7.8.2)$$

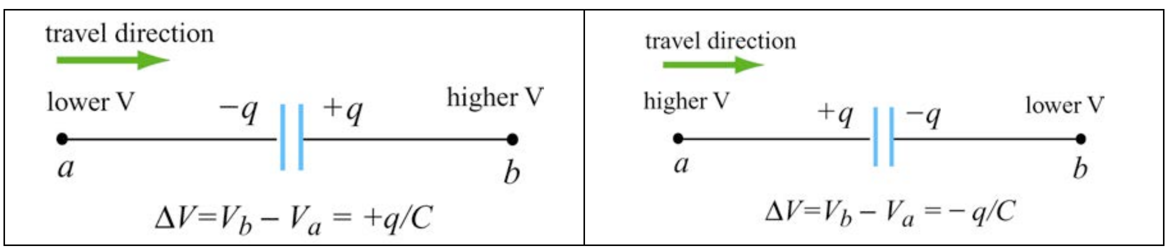


Figure 7.8.2 Rules for determining potential difference across capacitors.

We now use the loop rule that the sum of the potential differences around a closed loop is zero. We traverse the loop in the clockwise direction, and according the rules shown in Figure 7.8.2 for capacitors, $\Delta V_C = -q / C$. Therefore the loop rule becomes

$$0 = \varepsilon - I(t)R - q / C. \qquad (7.8.3)$$

The charge on the positive plate is increasing due to the addition charge that flows towards it,

$$I = +\frac{dq}{dt}, \qquad \text{(charging)}. \qquad (7.8.4)$$

Therefore Eq. (7.8.3) becomes

$$0 = \varepsilon - \frac{dq}{dt}R - \frac{q}{C}. \tag{7.8.5}$$

The current flow in the circuit will continue to decrease because the charge already present on *the* capacitor makes it harder to put more charge on the capacitor. Once the charge on the capacitor plates reaches its maximum value Q, the current in the circuit will drop to zero. This is evident by rewriting the loop law as

$$I(t)\,R = \varepsilon - \Delta V_C(t)\,. \qquad (7.8.6).$$

Thus, the charging capacitor satisfies a first order differential equation that relates the rate of change of charge to the charge on the capacitor according to

$$\frac{dq}{dt} = \frac{1}{R}\left(\varepsilon - \frac{q}{C}\right). \tag{7.8.7}$$

This equation can be solved by the method of separation of variables. The first step is to separate terms involving charge and time, (this means putting terms involving *dq* and *q* on one side of the equality sign and terms involving dt on the other side),

$$\frac{dq}{\left(\varepsilon - \dfrac{q}{C}\right)} = \frac{1}{R}dt \quad \Rightarrow \quad \frac{dq}{q - C\varepsilon} = -\frac{1}{RC}dt\,. \qquad (7.8.8).$$

Now we can integrate both sides of the above equation,

$$\int_0^q \frac{dq'}{q' - C\varepsilon} = -\frac{1}{RC}\int_0^t dt'\,. \tag{7.8.9}$$

which yields

$$\ln\left(\frac{q - C\varepsilon}{-C\varepsilon}\right) = -\frac{t}{RC}\,. \tag{7.8.10}$$

Exponentiate both sides of Eq. (7.8.10) using the fact that $\exp(\ln x) = x$ to yield

$$q(t) = C\varepsilon(1 - e^{-t/RC}) = Q(1 - e^{-t/RC})\,, \tag{7.8.11}$$

where $Q = C\varepsilon$ is the maximum amount of charge stored on the plates. The time dependence of $q(t)$ is plotted in Figure 7.8.3 below:

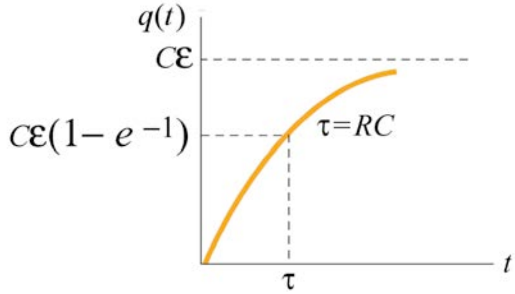


Figure 7.8.3 Plot of charge $q(t)$ on capacitor as a function of time during the charging process.

Once we know the charge on the capacitor we also can determine the potential difference across the capacitor,

$$\Delta V_C(t) = \frac{q(t)}{C} = \varepsilon(1 - e^{-t/RC}) . \qquad (7.8.12)$$

The graph of potential difference as a function of time has the same form as Figure 7.8.3. From the figure, we see that after a sufficiently long time the charge on the capacitor approaches the value

$$q(t = \infty) = C\varepsilon = Q . \qquad (7.8.13).$$

At that time, the potential difference across the capacitor is equal to the source emf and the charging process effectively ends,

$$\Delta V_C = \frac{q(t = \infty)}{C} = \frac{Q}{C} = \varepsilon . \qquad (7.8.14).$$

The current that flows in the circuit is equal to the derivative in time of the charge,

$$I(t) = \frac{dq}{dt} = \left(\frac{\varepsilon}{R}\right) e^{-t/RC} = I_0\, e^{-t/RC} . \qquad (7.8.15).$$

The coefficient in front of the exponential, $I_0 = \varepsilon / R$ **,** is equal to the initial current that flows in the circuit when the switch was closed at $t = 0$. The graph of current as a function of time is shown in Figure 7.8.4.

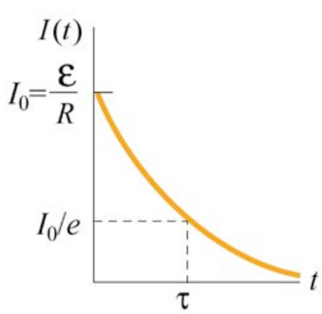


Figure 7.8.4 Plot of current $I(t)$ as a function of time during the charging process.

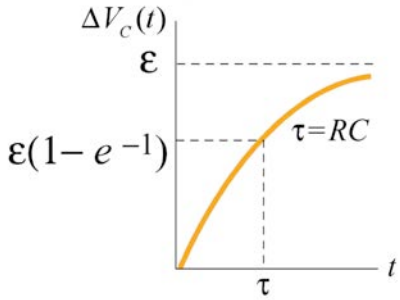


Figure 7.8.5 Plot of potential difference $\Delta V_C(t)$ across capacitor as a function of time during the charging process.

The current in the charging circuit decreases exponentially in time, $I(t) = I_0\, e^{-t/RC}$. This function is often written as $I(t) = I_0\, e^{-t/\tau}$ where $\tau = RC$ is called the *time constant.* The SI units of τ are seconds, as can be seen from the dimensional analysis:

$$[\text{Ù}][\text{F}]=([\text{V}]/[\text{A}])([\text{C}]/[\text{V}])=[\text{C}]/[\text{A}]=[\text{C}]/([\text{C}]/[\text{s}])=[\text{s}].$$

The time constant τ is a measure of the decay time for the exponential function. This decay rate satisfies the following property,

$$I(t+\tau) = I(t)\, e^{-1}, \tag{7.8.16}$$

which shows that after one time constant τ has elapsed, the current falls off by a factor of $e^{-1} = 0.368$, as indicated in Figure 7.8.4. Similarly, the potential difference across the capacitor, Figure 7.8.5, can also be expressed in terms of the time constant τ,

$$\Delta V_C(t) = \varepsilon(1 - e^{-t/\tau}). \tag{7.8.17}$$

Notice that initially at time $t = 0$, $\Delta V_C(t = 0) = 0$. After one time constant τ has elapsed, the potential difference across the capacitor plates has increased by a factor $(1 - e^{-1}) = 0.632$ of its final value,

$$\Delta V_C(\tau) = \varepsilon(1 - e^{-1}) = 0.632\,\varepsilon. \tag{7.8.18}$$

7.8.2 Discharging a Capacitor

Suppose initially the capacitor has been charged to some value *Q*. For $t < 0$, the switch is open and the potential difference across the capacitor is given by $\Delta V_C = Q/C$. The potential difference across the resistor is zero because there is no current through it, $I = 0$. Now suppose at $t = 0$ the switch is closed (Figure 7.8.6). The capacitor will begin to discharge.

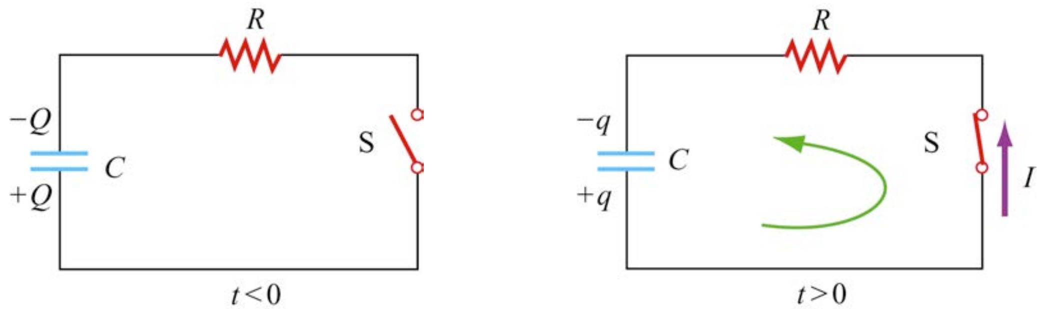


Figure 7.8.6 Discharging the RC circuit

The charged capacitor is now acting like a voltage source to drive current around the circuit, however the force on the charges is due to the electric fields. When the capacitor discharges (electrons flow from the negative plate through the wire to the positive plate), the potential difference across the capacitor decreases. The capacitor is losing strength as a voltage source. Traverse the loop counterclockwise, applying $\Delta V_C = +q / C$, (Figure 7.8.2) then the loop rule that describes the discharging process is given by

$$\frac{q}{C} - IR = 0. \qquad (7.8.19)$$

The charge on the positive plate is decreasing as charge flows away from the positive plate,

$$I = -\frac{dq}{dt}. \qquad (7.8.20)$$

Thus, charge satisfies a first order differential equation:

$$\frac{q}{C} + R\frac{dq}{dt} = 0. \qquad (7.8.21).$$

This equation can also be integrated by the method of separation of variables

$$\frac{dq}{q} = -\frac{1}{RC}dt, \qquad (7.8.22)$$

which yields

$$\int_Q^q \frac{dq'}{q'} = -\frac{1}{RC}\int_0^t dt' \quad \Rightarrow \quad \ln\left(\frac{q}{Q}\right) = -\frac{t}{RC}. \qquad (7.8.23)$$

After exponentiation,

$$\boxed{q(t) = Q e^{-t/RC}}. \qquad (7.8.24)$$

The potential difference across the capacitor is then

$$\Delta V_C(t) = \frac{q(t)}{C} = \left(\frac{Q}{C}\right)e^{-t/RC}; \qquad (7.8.25)$$

A plot of potential difference across the capacitor vs. time for the discharging capacitor is shown in Figure 7.8.7.

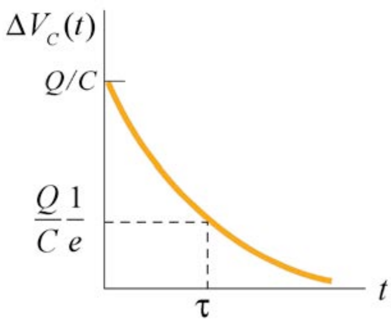


Figure 7.8.7 Plot of potential difference $\Delta V_C(t)$ across the capacitor as a function of time for discharging capacitor.

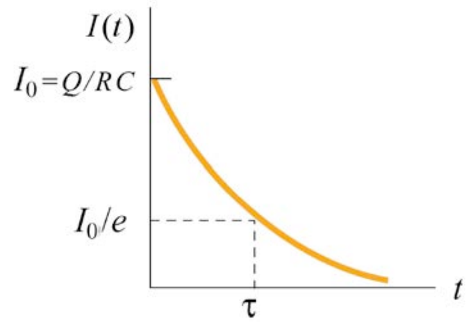


Figure 7.8.8 Plot of current $I(t)$ as a function of time for discharging capacitor.

The current also exponentially decays in the circuit as can be seen by differentiating the charge on the capacitor

$$I(t) = -\frac{dq}{dt} = \left(\frac{Q}{RC}\right)e^{-t/RC}. \qquad (7.8.26)$$

A plot of the current flowing in the circuit as a function of time also has the same form as the potential difference graph depicted in Figure 7.8.8.

7.9 Summary

- The equivalent resistance of a set of resistors connected in series:

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \cdots = \sum_{i=1}^{N} R_i .$$

- The equivalent resistance of a set of resistors connected in parallel:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{i=1}^{N} \frac{1}{R_i}.$$

- Kirchhoff's rules:

(1) The sum of the currents directed into a node (junction of branches) is equal to the sum of the currents directed out of the node:

$$\sum I_{\text{in}} = \sum I_{\text{out}} .$$

(2) The algebraic sum of the changes in electric potential in a closed-circuit loop is zero.

$$\sum_{\text{closed loop}} \Delta V = 0 .$$

- Power, or rate at which energy is delivered to the resistor is

$$P = I\Delta V = I^2 R = \frac{(\Delta V)^2}{R} .$$

- Power, or rate at which energy is delivered from source of emf

$$P = I\varepsilon .$$

- The equivalent capacitance of capacitors connected in parallel and in series are

$$C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots \quad \text{(parallel)},$$

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \quad \text{(series)} .$$

- In a charging capacitor, the charge and the current as a function of time are

$$q(t) = Q(1 - e^{-t/RC}), \qquad I(t) = \frac{\varepsilon}{R} e^{-t/RC} .$$

- In a discharging capacitor, the charge and the current as a function of time are

$$q(t) = Q e^{-t/RC}, \quad I(t) = \frac{Q}{RC} e^{-t/RC} .$$

7.10 Problem-Solving Strategy: Applying Kirchhoff's Rules

In this chapter we have seen how Kirchhoff's rules can be used to analyze multi-loop circuits. The steps are summarized below:

(1) Draw a circuit diagram, and label all the quantities, both known and unknown. The

number of unknown quantities is equal to the number of linearly independent equations we must look for.

(2) Assign a direction to the current in each branch of the circuit. (If the actual direction is opposite to what you have assumed, your result at the end will be a negative number.) Let B equal to the number of branches.

(3) If there are M junctions, apply the junction rule to $M-1$ junctions. Applying the junction rule to the last junction will not yield a new independent relationship among the currents.

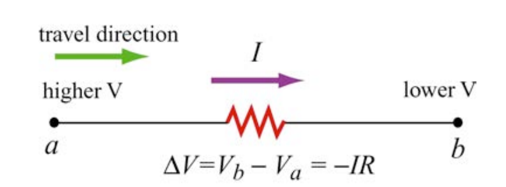
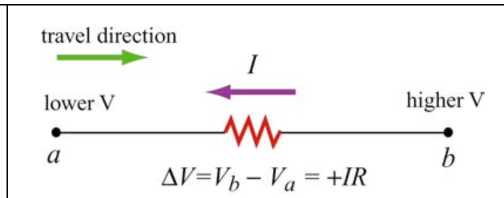
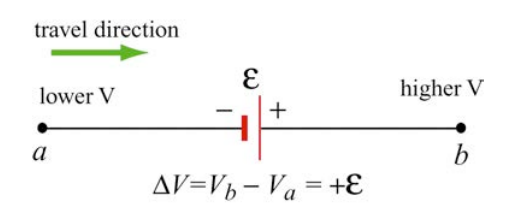
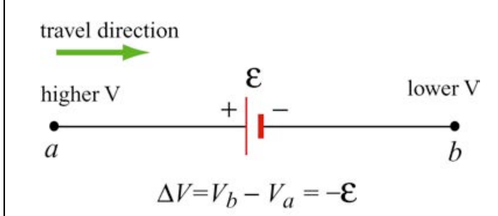
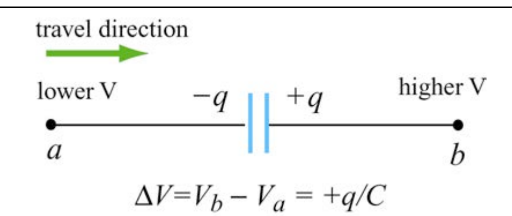
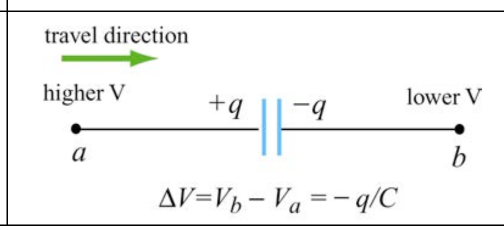
(4) If there are N loops, apply the loop rule to $N-1$ loops. Applying the loop rule to the last loop will not yield a new independent relationship among the loop equations.

Then

$$B=(M-1)+(N-1) \tag{7.10.1}$$

For example, if there are three branches with three unknown currents than there must be two junctions and three loops. Therefore we must write down one junction equation and two loop equations to total three linearly independent equations in order to have a unique solution.

Traverse the loops using the convention below for electric potential difference ΔV across each circuit element:

resistor		
emf source		
capacitor		

The same equation is obtained whether the closed loop is traversed clockwise or counterclockwise.

(5) Solve the simultaneous equations to obtain the solutions for the unknown quantities.

As an example of illustrating how the above procedures are executed, let's analyze the circuit shown in Figure 7.10.1.

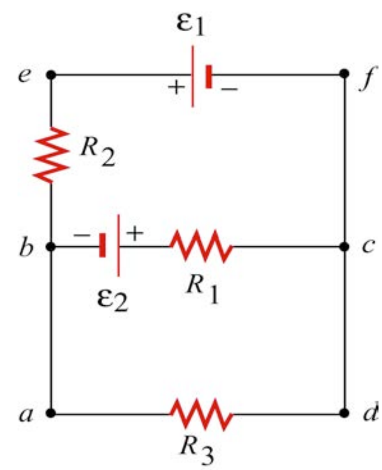


Figure 7.10.1 A multi-loop circuit.

Suppose the emf sources ε_1 and ε_2, and the resistances R_1, R_2 and R_3 are all given, and we would like to find the currents through each resistor, using the methodology outlined above.

(1) The unknown quantities are the three currents I_1, I_2 and I_3, associated with the three resistors. Therefore, to solve the system, we must look for three independent equations.

(2) The directions for the three currents are arbitrarily assigned, as indicated in Figure 7.10.2.

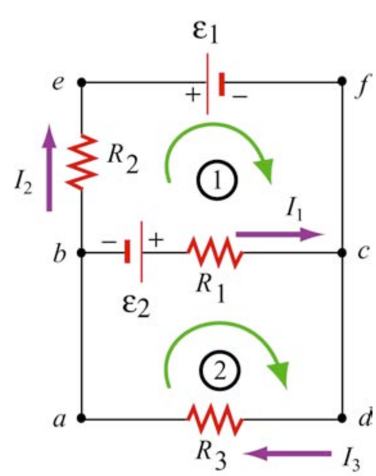


Figure 7.10.2

(3) Applying Kirchhoff's junction rule to node *b* yields

$$I_1 + I_2 = I_3 ,$$

because I_1 and I_2 are leaving the node while I_3 is entering the node. The same equation is obtained if we consider node *c*.

(4) The other two equations can be obtained by using the loop rule, which states that the sum of the potential difference across all elements in a closed circuit loop is zero. Traversing the first loop *befcb* in the clockwise direction yields

$$-I_2 R_2 - \varepsilon_1 + I_1 R_1 - \varepsilon_2 = 0 .$$

Similarly, traversing the second loop *abcda* clockwise gives

$$\varepsilon_2 - I_1 R_1 - I_3 R_3 = 0 .$$

Note however, that one may also consider the big loop *abefcda*. This leads to

$$-I_2 R_2 - \varepsilon_1 - I_3 R_3 = 0 .$$

However, the equation is not linearly independent of the other two loop equations since it is simply the sum of those equations.

(5) The solutions to the above three equations are given by, after tedious but straightforward algebra,

$$I_1 = \frac{\varepsilon_1 R_3 + \varepsilon_2 R_3 + \varepsilon_2 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3},$$

$$I_2 = -\frac{\varepsilon_1 R_1 + \varepsilon_1 R_3 + \varepsilon_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3},$$

$$I_3 = \frac{\varepsilon_2 R_2 - \varepsilon_1 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3}.$$

Note that I_2 is a negative quantity. This simply indicates that the direction of I_2 is opposite of what we have initially assumed.

7.11 Solved Problems

7.11.1 Equivalent Resistance

Consider the circuit shown in Figure 7.11.1. For a given resistance R_0, what must be the value of R_1 so that the equivalent resistance between the terminals is equal to R_0?

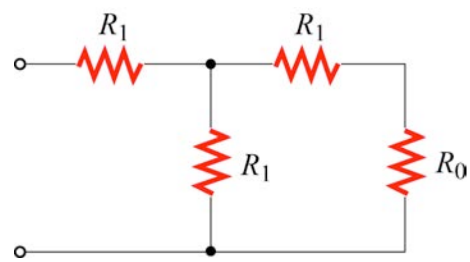


Figure 7.11.1 Resistor network

Solution: We first add resistances $R' = R_0 + R_1$ because those resistors are in series to obtain our first equivalent network (Figure 7.11.2(a).)

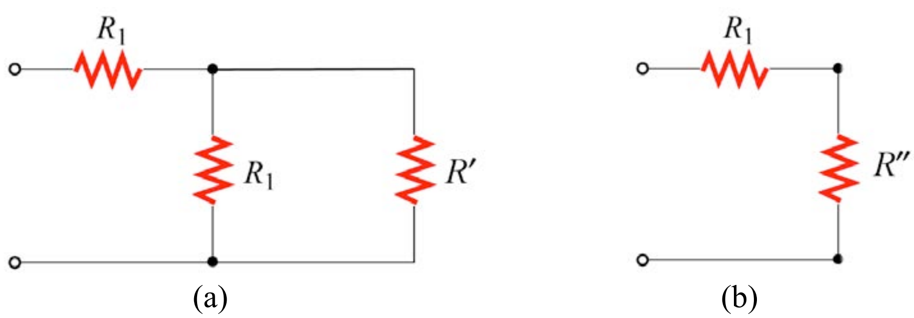


Figure 7.11.2 (a) and (b) Equivalent networks

Because the two resistors in the equivalent network are in parallel the inverse of the equivalent resistance, R'', is obtaining by adding the resistances inversely,

$$\frac{1}{R''} = \frac{1}{R_1} + \frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_0 + R_1} = \frac{R_0 + 2R_1}{R_1(R_0 + R_1)} .$$

Therefore after some algebra,

$$R'' = \frac{R_1(R_0 + R_1)}{R_0 + 2R_1} .$$

The second equivalent network is shown in Figure 7.11.2(b). Because the new equivalent resistor with resistance R'' is in series with the fourth resistor with resistance R_1, the equivalent resistance of the entire configuration becomes

$$R_{\text{eq}} = R_1 + R'' = R_1 + \frac{R_1(R_0 + R_1)}{R_0 + 2R_1} = \frac{3R_1^{\,2} + 2R_1R_0}{R_0 + 2R_1}.$$

If $R_{\text{eq}} = R_0$, then

$$R_0(R_0 + 2R_1) = 3R_1^{\,2} + 2R_1R_0 \;\Rightarrow\; R_0^{\,2} = 3R_1^{\,2} \Rightarrow R_1 = \frac{R_0}{\sqrt{3}}.$$

7.11.2 Variable Resistance

Show that, if a battery of fixed emf ε and internal resistance r is connected to a variable external resistance R, the maximum power is delivered to the external resistor when $R = r$.

Solution: Using Kirchhoff's rule,

$$\varepsilon = I(R + r),$$

which implies

$$I = \frac{\varepsilon}{R + r}.$$

The power dissipated in the resistor is equal to

$$P = I^2 R = \frac{\varepsilon^2}{(R + r)^2} R.$$

To find the value of R which gives out the maximum power, we differentiate P with respect to R and set the derivative equal to 0,

$$\frac{dP}{dR} = \varepsilon^2 \left[\frac{1}{(R + r)^2} - \frac{2R}{(R + r)^2} \right] = \varepsilon^2 \frac{r - R}{(R + r)^3} = 0,$$

which implies

$$R = r.$$

This is an example of "impedance matching," in which the variable resistance R is adjusted so that the power delivered to it is maximized. The behavior of P as a function of R is depicted in Figure 7.11.2.

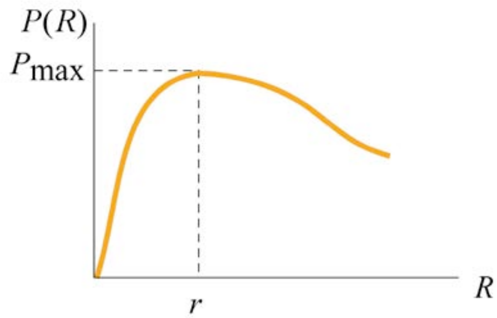


Figure 7.11.2 Plot of power dissipated $P(R)$ as a function of R.

7.11.3 RC Circuit

In the circuit in Figure 7.11.3, suppose the switch has been open for a very long time. At time $t = 0$, it is suddenly closed.

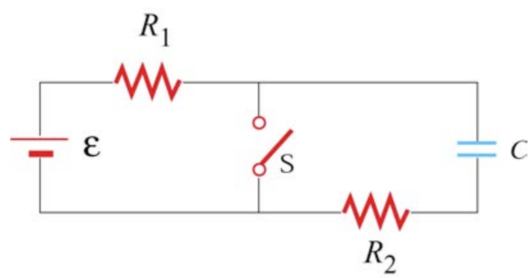


Figure 7.11.3

(a) What is the time constant before the switch is closed?

(b) What is the time constant after the switch is closed?

(c) Find the current through the switch as a function of time after the switch is closed.

Solutions: (a) Before the switch is closed, the two resistors with resistances R_1 and R_2 are in series with the capacitor. Since the equivalent resistance is $R_{eq} = R_1 + R_2$, the time constant is given by

$$\tau = R_{eq}C = (R_1 + R_2)C .$$

The amount of charge stored in the capacitor is

$$q(t) = C\varepsilon(1 - e^{-t/\tau}) .$$

(b) After the switch is closed, the closed loop on the right becomes a decaying *RC* circuit with time constant $\tau' = R_2 C$. Charge begins to decay according to

$$q'(t) = C\varepsilon e^{-t/\tau'} .$$

(c) The current passing through the switch consists of two sources: the steady current I_1 from the left circuit, and the decaying current I_2 from the RC circuit. The currents are given by

$$I_1 = \frac{\varepsilon}{R_1}$$

$$I'(t) = \frac{dq'}{dt} = -\frac{C\varepsilon}{\tau'} e^{-t/\tau'} = -\frac{\varepsilon}{R_2} e^{-t/R_2C} .$$

The negative sign in $I'(t)$ indicates that the direction of flow is opposite of the charging process. Thus, since both I_1 and I' move downward across the switch, the total current is

$$I(t) = I_1 + I'(t) = \frac{\varepsilon}{R_1} + \frac{\varepsilon}{R_2} e^{-t/R_2C} .$$

7.11.4 Parallel vs. Series Connections

Figure 7.11.4 show two resistors with resistances R_1 and R_2 connected in parallel and in series. The battery has emf ε .

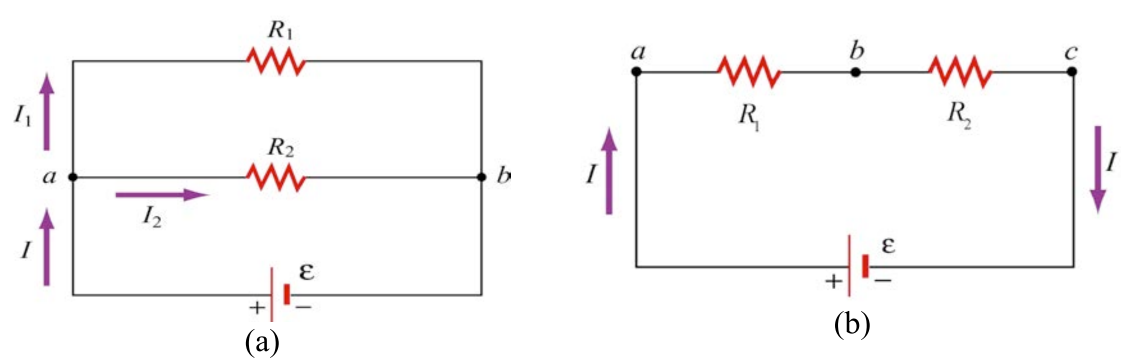


Figure 7.11.4 (a) parallel, (b) series

Suppose R_1 and R_2 are connected in parallel (Figure 7.11.4(a)).

(a) Find the power delivered to each resistor.

(b) Show that the sum of the power used by each resistor is equal to the power supplied by the battery.

Suppose R_1 and R_2 are now connected in series.

(c) Find the power delivered to each resistor.

(d) Show that the sum of the power used by each resistor is equal to the power supplied by the battery.

(e) Which configuration, parallel or series, uses more power?

Solutions:

(a) When two resistors are connected in parallel, the current through each resistor is

$$I_1 = \frac{\varepsilon}{R_1}, \quad I_2 = \frac{\varepsilon}{R_2} \quad ,$$

and the power delivered to each resistor is given by

$$P_1 = I_1^2 R_1 = \frac{\varepsilon^2}{R_1}, \quad P_2 = I_2^2 R_2 = \frac{\varepsilon^2}{R_2}.$$

The results indicate that the smaller the resistance, the greater the amount of power delivered. If the loads are the light bulbs, then the one with smaller resistance will be brighter since more power is delivered to it.

(b) The total power delivered to the two resistors is

$$P_R = P_1 + P_2 = \frac{\varepsilon^2}{R_1} + \frac{\varepsilon^2}{R_2} = \frac{\varepsilon^2}{R_{\text{eq}}},$$

where

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \quad \Rightarrow R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}$$

is the equivalent resistance of the circuit. On the other hand, the total power supplied by the battery is $P_\varepsilon = I\varepsilon$, where $I = I_1 + I_2$, as seen from Figure 7.11.4(a). Thus,

$$P_\varepsilon = I_1\varepsilon + I_2\varepsilon = \left(\frac{\varepsilon}{R_1}\right)\varepsilon + \left(\frac{\varepsilon}{R_2}\right)\varepsilon = \frac{\varepsilon^2}{R_1} + \frac{\varepsilon^2}{R_2} = \frac{\varepsilon^2}{R_{\text{eq}}} = P_R,$$

as required by energy conservation.

(c) When the two resistors are connected in series (Figure 7.11.4(b)), the equivalent resistance becomes

$$R'_{\text{eq}} = R_1 + R_2 \, ,$$

and the currents through the resistors are

$$I_1 = I_2 = I = \frac{\varepsilon}{R_1 + R_2} \, .$$

Therefore, the power delivered to each resistor is

$$P_1 = I_1^2 R_1 = \left(\frac{\varepsilon}{R_1 + R_2} \right)^2 R_1 \, , \quad P_2 = I_2^2 R_2 = \left(\frac{\varepsilon}{R_1 + R_2} \right)^2 R_2 \, .$$

Contrary to what we have seen in the parallel case, when connected in series, the greater the resistance, the greater the fraction of the power delivered. Once again, if the loads are light bulbs, the one with greater resistance will be brighter.

(d) The total power delivered to the resistors is

$$P_R' = P_1 + P_2 = \left(\frac{\varepsilon}{R_1 + R_2} \right)^2 R_1 + \left(\frac{\varepsilon}{R_1 + R_2} \right)^2 R_2 = \frac{\varepsilon^2}{R_1 + R_2} = \frac{\varepsilon^2}{R'_{\text{eq}}} \, .$$

On the other hand, the power supplied by the battery is

$$P_\varepsilon' = I\varepsilon = \left(\frac{\varepsilon}{R_1 + R_2} \right) \varepsilon = \frac{\varepsilon^2}{R_1 + R_2} = \frac{\varepsilon^2}{R'_{\text{eq}}} \, .$$

Again, we see that $P_\varepsilon' = P_R'$, as required by energy conservation.

(e) Comparing the results obtained in (b) and (d), we see that

$$P_\varepsilon = \frac{\varepsilon^2}{R_1} + \frac{\varepsilon^2}{R_2} > \frac{\varepsilon^2}{R_1 + R_2} = P_\varepsilon' \, ,$$

which means that the parallel connection uses more power. The equivalent resistance of two resistors connected in parallel is always smaller than that connected in series.

7.11.5 Resistor Network

Consider a cube that has identical resistors with resistance R along each edge, as shown in Figure 7.11.5.

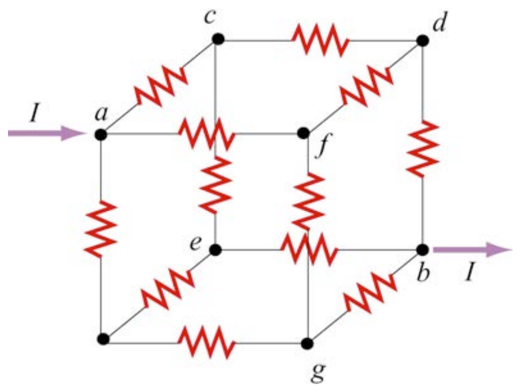


Figure 7.11.5 Resistor network

Find the equivalent resistance between points a and b .

Solution: From symmetry arguments, the current which enters a must split evenly, with $I/3$ going to each branch. At the next junction, say c, $I/3$ must further split evenly with $I/6$ going through the two paths ce and cd. The current going through the resistor in db is the sum of the currents from fd and cd , therefore

$$I/6 + I/6 = I/3 .$$

Thus, the potential difference between a and b can be obtained as

$$V_{ab} = V_{ac} + V_{cd} + V_{db} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}IR .$$

Hence the equivalent resistance is

$$R_{\text{eq}} = \frac{5}{6}R .$$

7.11.6 Equivalent Capacitance

Consider the configuration shown in Figure 7.11.6. Find the equivalent capacitance, assuming that all the capacitors have the same capacitance C.

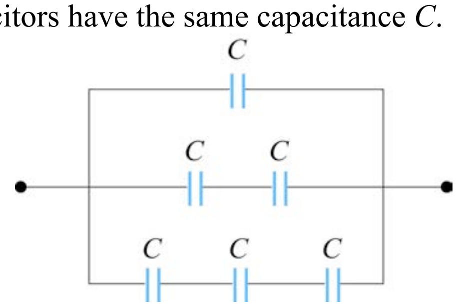


Figure 7.11.6 Combination of Capacitors

Solution: For capacitors that are connected in series, the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots = \sum_i \frac{1}{C_i} \qquad \text{(series)}\,.$$

On the other hand, for capacitors that are connected in parallel, the equivalent capacitance is

$$C_{\text{eq}} = C_1 + C_2 + \cdots = \sum_i C_i \qquad \text{(parallel)}\,.$$

Using the above formula for series connection, the equivalent configuration is shown in Figure 7.11.7.

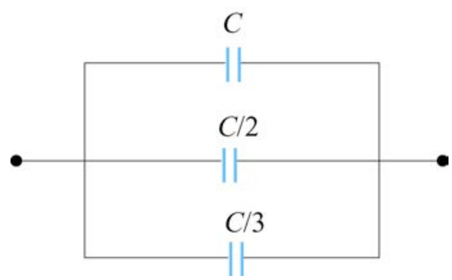


Figure 7.11.7

Now we have three capacitors connected in parallel. The equivalent capacitance is given by

$$C_{\text{eq}} = C\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{6}C\,.$$

7.12 Conceptual Questions

1. Given three resistors of resistances R_1, R_2 and R_3, how should they be connected to (a) maximize (b) minimize the equivalent resistance?

2. Why do the headlights on the car become dim when the car is starting?

3. Does the resistor in an RC circuit affect the maximum amount of charge that can be stored in a capacitor? Explain.

4. Can one construct a circuit such that the potential difference across the terminals of the battery is zero? Explain.

5. Two conductors A and B of the same length and radius are connected across the same potential difference. The resistance of conductor A is twice that of B. To which conductor is more power delivered?

7.13 Additional Problems

7.13.1 Resistive Circuits

Consider two identical batteries of emf ε and internal resistance r. They may be connected in series or in parallel and are used to establish a current in resistance R as shown in Figure 7.13.1.

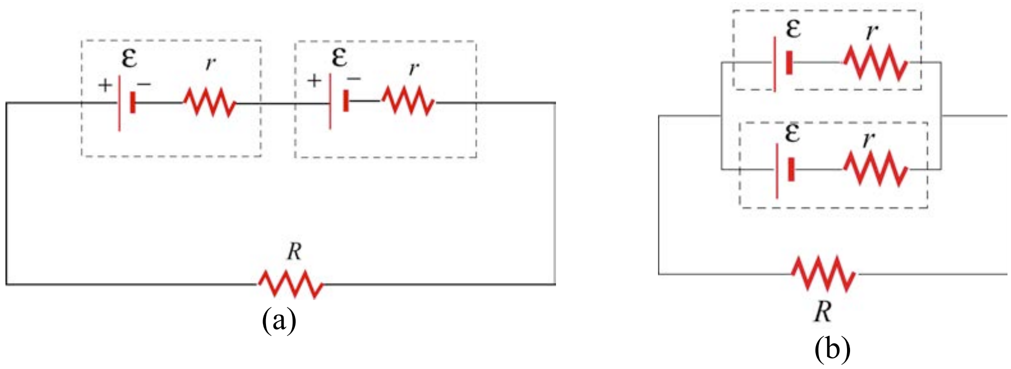


Figure 7.13.1 Two batteries connected in (a) series, and (b) parallel.

(a) Derive an expression for the current in R for the series connection shown in Figure 7.13.1(a). Be sure to indicate the current on the sketch (to establish a sign convention for the direction) and apply Kirchhoff's loop rule.

(b) Find the current for the parallel connection shown in Figure 7.13.1(b).

(c) For what relative values of r and R would the currents in the two configurations be the same; be larger in Figure 7.13.1(a); be larger in 7.13.1(b)?

7.13.2 Multi-loop Circuit

Consider the circuit shown in Figure 7.13.2. Neglecting the internal resistance of the batteries, calculate the currents through each of the three resistors.

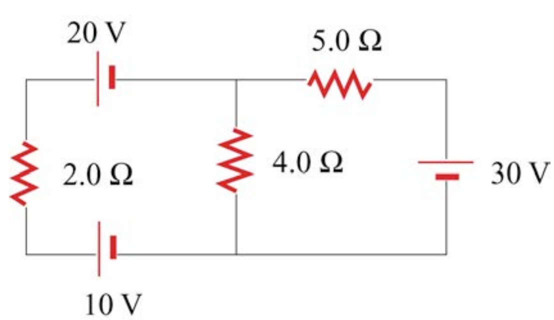


Figure 7.13.2

7.13.3 Power Delivered to the Resistors

Consider the circuit shown in Figure 7.13.3. Find the power delivered to each resistor.

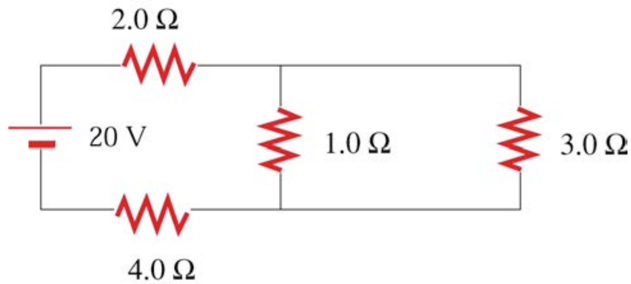


Figure 7.13.3

7.13.4 Resistor Network

Consider an infinite network of resistors of resistances R_0 and R_1 shown in Figure 7.13.4. Show that the equivalent resistance of this network is

$$R_{eq} = R_1 + \sqrt{R_1^2 + 2R_1R_0}\ .$$

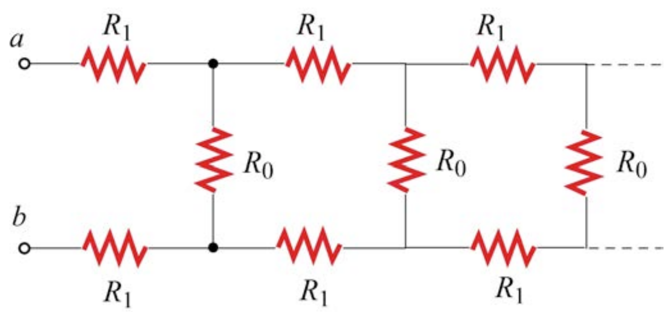


Figure 7.13.4

7.13.5 RC Circuit

Consider the circuit shown in Figure 7.13.5. Let $\varepsilon = 40\ \text{V}$, $R_1 = 8.0\ \Omega$, $R_2 = 6.0\ \Omega$, $R_3 = 4.0\ \Omega$ and $C = 4.0\ \mu\text{F}$. The capacitor is initially uncharged.

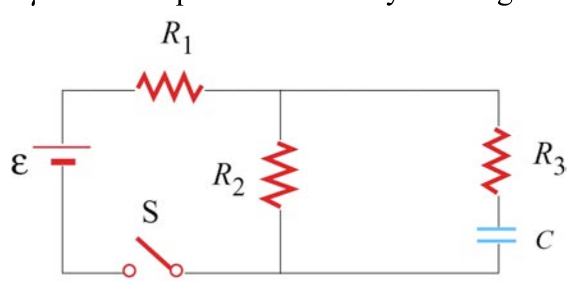


Figure 7.13.5

At $t = 0$, the switch is closed.

(a) Find the current through each resistor immediately after the switch is closed.

(b) Find the final charge on the capacitor.

7.13.6 Resistors in Series and Parallel

A circuit containing five resistors and a 12 V battery is shown in Figure 7.13.6. Find the potential drop across the 5Ω resistor. [Ans. 7.5 V].

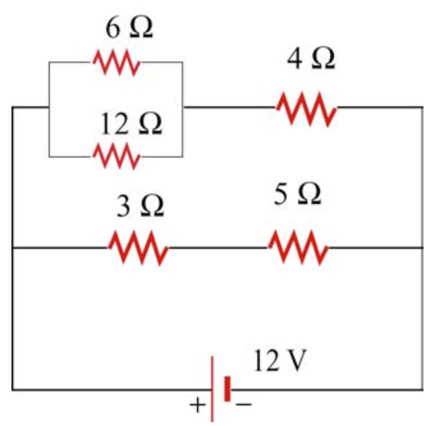


Figure 7.13.6

7.13.7 Capacitors in Series and in Parallel

A 12-Volt battery charges the four capacitors shown in Figure 7.13.7.

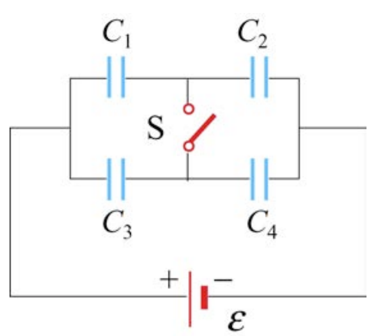


Figure 7.13.7

Let C_1 = 1 μF, C_2 = 2 μF, C_3 = 3 μF, and C_4 = 4 μF.

(a) What is the equivalent capacitance of the group C_1 and C_2 if switch S is open (as shown)?

(b) What is the charge on *each* of the four capacitors if switch S is open?

(c) What is the charge on each of the four capacitors if switch S is closed?

7.13.8 Power Loss and Ohm's Law

A 1500 W radiant heater is constructed to operate at 115 V.

(a) What will be the current in the heater? [Ans. ~10 A]

(b) What is the resistance of the heating coil? [Ans. ~10 Ω]

(c) How many kilocalories are generated in one hour by the heater? (1 Calorie = 4.18 J)

7.13.9 Power, Current, and Potential difference

A 100-W light bulb is plugged into a standard 120-V outlet. (a) How much does it cost per month (31 days) to leave the light turned on? Assume electricity costs 6 cents per $\mathrm{kW\cdot h}$. (b) What is the resistance of the bulb? (c) What is the current in the bulb? [Ans. (a) $4.46; (b) 144 Ω; (c) 0.833 A].