Chapter 4

Electric Potential

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4.1 Potential and Potential Energy

In the introductory mechanics course, we have seen that force on a particle of mass m located at a distance r from Earth's center due to the gravitational interaction between the particle and the Earth obeys an inverse-square law:

$$\vec{\mathbf{F}}_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}}\,, \tag{4.1.1}$$

where $G = 6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}$ is the gravitational constant and $\hat{\mathbf{r}}$ is a unit vector pointing radially outward from the Earth. The Earth is assumed to be a uniform sphere of mass M. The corresponding gravitation field $\vec{\mathbf{g}}$, defined as the gravitation force per unit mass, is given by

$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}_g}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}}\,. \tag{4.1.2}$$

Notice that $\vec{\mathbf{g}}$ is a function of M, the mass that creates the field, and r, the distance from the center of the Earth.

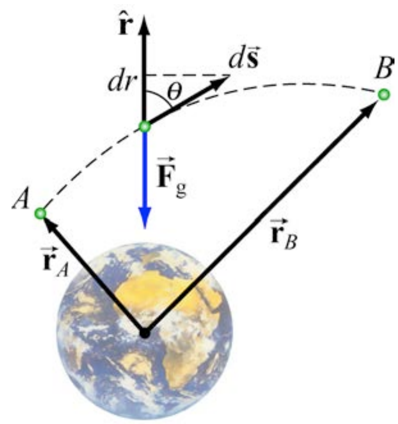


Figure 4.1.1

Consider moving a particle of mass m under the influence of gravity (Figure 4.1.1). The work done by gravity in moving m from A to B is

$$W_G = \int \vec{\mathbf{F}}_G\cdot d\vec{\mathbf{s}} = \int_{r_A}^{r_B}\left(-\frac{GMm}{r^2}\right)dr = \left[\frac{GMm}{r}\right]_{r_A}^{r_B} = GMm\left(\frac{1}{r_B}-\frac{1}{r_A}\right). \tag{4.1.3}$$

The result shows that W_G is independent of the path taken; it depends only on the endpoints *A* and *B*.

Near Earth's surface, the gravitational field $\vec{\mathbf{g}}$ is approximately constant, with a magnitude $g = GM / r_E^{\,2} \approx 9.8\,\text{m/s}^2$, where r_E is the radius of Earth. The work done by gravity in moving an object from height y_A to y_B (Figure 4.1.2) is

$$W_g = \int \vec{\mathbf{F}}_g \cdot d\vec{\mathbf{s}} = -\int_{y_A}^{y_B} mg\,dy = -mg(y_B - y_A)\,. \tag{4.1.4}$$

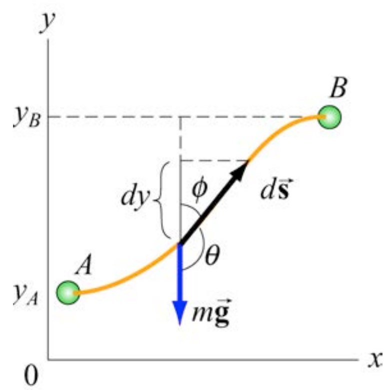


Figure 4.1.2 Moving an object from *A* to *B*.

The result again is independent of the path, and is only a function of the change in vertical height $y_B - y_A$.

In the examples above, if the object returns to its starting point, then the work done by the gravitation force on the object is zero along this *closed path*. Any force that satisfies this property for all closed paths is called a *conservative force*:

$$\oint_{\text{all closed paths}} \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}} = 0 \qquad \text{(conservative force).} \tag{4.1.5}$$

When dealing with a conservative force, it is often convenient to introduce the concept of change in potential energy function, $\Delta U = U_B - U_A$ between any two points in space, *A* and *B,*

$$\Delta U = U_B - U_A = -\int_A^B \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}} = -W \tag{4.1.6}$$

where W is the work done by the force on the object. In the case of gravity, $W = W_g$ and from Eq. ((4.1.3)), the change in potential energy can be written as

$$U_G(r_B) - U_G(r_A) = -GMm\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{4.1.7}$$

It is often convenient to choose a reference point P where $U_G(r_P)$ is equal to zero. In the gravitational case, we choose infinity to be the reference point, with $U_G(r_P = \infty) = 0$. Therefore the change in potential energy when two objects start off an infinite distance apart and end up a distance r apart is given by

$$U_G(r) - U_G(\infty) = -GMm\left(\frac{1}{r} - \frac{1}{\infty}\right) = -\frac{GMm}{r}\,. \tag{4.1.8}$$

Thus we can define a potential energy function

$$U_G(r) = -\frac{GMm}{r}\,, \qquad U(\infty) = 0\,. \tag{4.1.9}$$

When one object is much more massive for example the Earth and a satellite, then the scalar quantity $U_G(r)$, with units of energy, corresponds to the negative of the work done by the gravitation force on the satellite as it moves from an infinite distance away to a distance r from the center of the Earth. The value of $U_G(r)$ depends on the choice that $U_G(r_P = \infty) = 0$. However, the potential energy difference $U_G(r_B) - U_G(r_A)$ between two points is independent of the choice of reference point and by definition corresponds to a physical quantity, the negative of the work done.

Near Earth's surface, where the gravitation field $\vec{\mathbf{g}}$ is approximately constant, as an object moves from the ground to a height h above the ground, the change in potential energy is $\Delta U_g = +mgh$, and the work done by gravity is $W_g = -mgh$.

Let's again consider a gravitation field $\vec{\mathbf{g}}$. Let's define the change in potential energy per mass between points A and B by

$$\Delta V_G \equiv V_G(r_B) - V_G(r_A) = \frac{U_G(r_B) - U_G(r_A)}{m} \equiv \frac{\Delta U_G}{m} \tag{4.1.10}$$

According to our definition,

$$\Delta V_G = -\int_A^B (\vec{\mathbf{F}}_G / m) \cdot d\,\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{g}} \cdot d\,\vec{\mathbf{s}} \tag{4.1.11}$$

ΔV_G is called the *gravitation potential difference.* The terminology is unfortunate because it is very easy to mix-up 'potential difference' with 'potential energy difference'. From Eq. (4.1.7), the gravitation potential difference between the points A and B is

$$\Delta V_G = -\int_A^B (\vec{\mathbf{F}}_G / m) \cdot d\,\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{g}} \cdot d\,\vec{\mathbf{s}} \tag{4.1.12}$$

Just like the gravitation field, the gravitation potential difference depends only on the M, the mass that creates the field, and r, the distance from the center of the Earth. Physically ΔV_G represents the negative of the work done per unit mass by gravity to move a particle from points A to B.

Our treatment of electrostatics will be similar to gravitation because the electrostatic force $\vec{\mathbf{F}}_e$ also obeys an inverse-square law. In addition, it is also conservative. In the presence of an electric field $\vec{\mathbf{E}}$, in analogy to the gravitational field $\vec{\mathbf{g}}$, we define the electric potential difference between two points A and B as

$$\Delta V_e = -\int_A^B (\vec{\mathbf{F}}_e / q_t) \cdot d\,\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}}\,, \tag{4.1.13}$$

where q_t is a test charge. The potential difference ΔV_e, which we will now denote just by ΔV, represents the negative of the work done per unit charge by the electrostatic force when a test charge q_t moves from points A to B. Again, electric potential difference should not be confused with electric potential energy. The two quantities are related as follows. Suppose an object with charge q is moved across a potential difference ΔV, then the change in the potential energy of the object is

$$\Delta U = q\Delta V\,. \tag{4.1.14}$$

The SI unit of electric potential is volt $[\mathrm{V}]$

$$1\,\text{volt} = 1\ \text{joule/coulomb}\ \ (1\ \text{V}=1\ \text{J/C})\,. \tag{4.1.15}$$

When dealing with systems at the atomic or molecular scale, a joule $[\mathrm{J}]$ often turns out to be too large as an energy unit. A more useful scale is electron volt $[\mathrm{eV}]$, which is defined as the energy an electron acquires (or loses) when moving through a potential difference of one volt:

$$1\,\mathrm{eV} = (1.6\times10^{-19}\,\mathrm{C})(1\,\mathrm{V}) = 1.6\times10^{-19}\,\mathrm{J}\,. \tag{4.1.16}$$

4.2 Electric Potential in a Uniform Field

Consider a charge $+q$ moving in the direction of a uniform electric field $\vec{\mathbf{E}} = E(-\hat{\mathbf{j}})$, as shown in Figure 4.2.1(a).

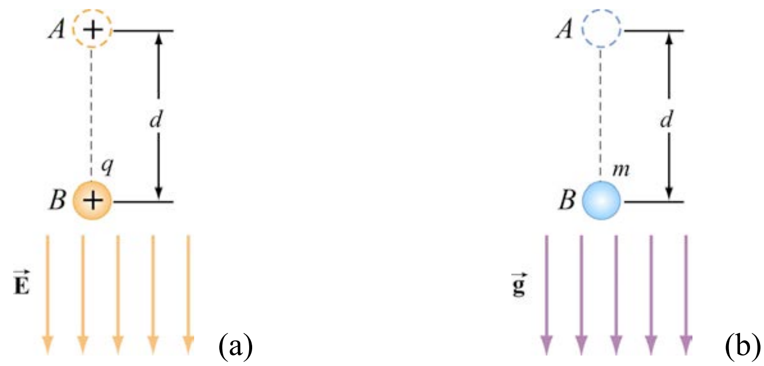


Figure 4.2.1 (a) A charge q moving in the direction of a constant electric field $\vec{\mathbf{E}}$. (b) A mass m moving in the direction of a constant gravitation field $\vec{\mathbf{g}}$.

Because the path taken is parallel to $\vec{\mathbf{E}}$, the electric potential difference between points A and B is given by

$$\Delta V = V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -E\int_A^B ds = -Ed < 0\,. \qquad (4.2.1)$$

Therefore point B is at a lower potential compared to point A. In fact, electric field lines always point from higher potential to lower. The change in potential energy is $\Delta U = U_B - U_A = -qEd$. Because $q > 0$, for this motion $\Delta U < 0$, the potential energy of a positive charge decreases as it moves along the direction of the electric field. The corresponding gravity analogy, depicted in Figure 4.2.1(b), is that a mass m loses potential energy ($\Delta U = -mgd$) as it moves in the direction of the gravitation field $\vec{\mathbf{g}}$.

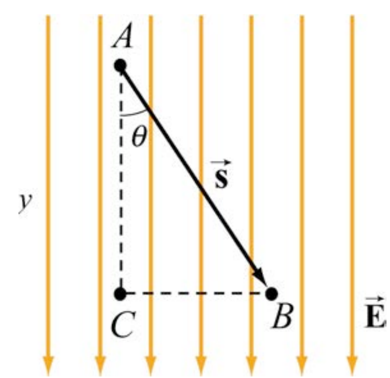


Figure 4.2.2 Potential difference in a uniform electric field

What happens if the path from A to B is not parallel to $\vec{\mathbf{E}}$, but instead at an angle θ, as shown in Figure 4.2.2? In that case, the potential difference becomes

$$\Delta V = V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\vec{\mathbf{E}} \cdot \vec{\mathbf{s}} = -Es\cos\theta = -Ey\,. \tag{4.2.2}$$

Note that y increases downward in Figure 4.2.2. Here we see once more that moving along the direction of the electric field $\vec{\mathbf{E}}$ leads to a lower electric potential. What would the change in potential be if the path were $A \to C \to B$? In this case, the potential difference consists of two contributions, one for each segment of the path:

$$\Delta V = \Delta V_{CA} + \Delta V_{BC}\,. \tag{4.2.3}$$

When moving from A to C, the change in potential is $\Delta V_{CA} = -Ey$. When moving from C to B, $\Delta V_{BC} = 0$ because the path is perpendicular to the direction of $\vec{\mathbf{E}}$. Thus, the same result is obtained irrespective of the path taken, consistent with the fact that $\vec{\mathbf{E}}$ is a conservative vector field.

For the path $A \to C \to B$, work is done by the field only along the segment AC that is parallel to the field lines. Points B and C are at the same electric potential, i.e., $V_B = V_C$. Because $\Delta U = q\Delta V$, this means that no work is required when moving the charge from B to C. In fact, all points along the straight line connecting B and C are on the same "equipotential line." A more complete discussion of equipotential will be given in Section 4.5.

4.3 Electric Potential due to Point Charges

Next, let's compute the potential difference between two points A and B due to a charge $+Q$. The electric field produced by Q is $\vec{\mathbf{E}} = (Q/4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector pointing radially away from the location of the charge.

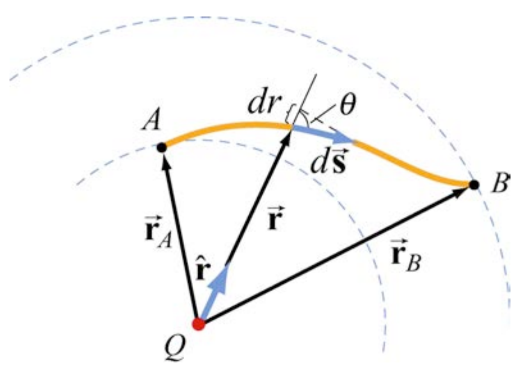


Figure 4.3.1 Potential difference between two points due to a point charge Q.

From Figure 4.3.1, we see that $\hat{\mathbf{r}} \cdot d\vec{\mathbf{s}} = ds\cos\theta = dr$, which gives

$$\Delta V = V_B - V_A = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \cdot d\,\vec{\mathbf{s}} = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right). \quad (4.3.1)$$

Once again, the potential difference ΔV depends only on the endpoints, independent of the choice of path taken. As in the case of gravity, only the difference in electrical potential is physically meaningful, and one may choose a reference point and set the potential there to be zero. In practice, it is often convenient to choose the reference point to be at infinity, so that the electric potential at a point *P* becomes

$$V_P = -\int_\infty^P \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}}\,, \qquad V(\infty) = 0\,. \quad (4.3.2)$$

With this choice of zero potential, we introduce an *electric potential function*, $V(r)$, where *r* is the distance from the point-like charged object with charge *Q*:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\,. \quad (4.3.3)$$

When more than one point charge is present, by applying the superposition principle, the electric potential is the sum of potentials due to individual charges:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i} = k_e \sum_i \frac{q_i}{r_i} \quad (4.3.4)$$

A summary of comparison between gravitation and electrostatics is tabulated below:

Gravity	Electrostatics
Mass m	Charge q
Gravitation force $\vec{\mathbf{F}}_G = -G\dfrac{Mm}{r^2}\hat{\mathbf{r}}$	Electric force $\vec{\mathbf{F}}_e = k_e\dfrac{Qq}{r^2}\hat{\mathbf{r}}$
Gravitation field $\vec{\mathbf{g}} = \vec{\mathbf{F}}_g / m$	Electric field $\vec{\mathbf{E}} = \vec{\mathbf{F}}_e / q$
Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_G \cdot d\vec{\mathbf{s}}$	Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_e \cdot d\vec{\mathbf{s}}$
Gravitational potential $\Delta V_G = -\int_A^B \vec{\mathbf{g}} \cdot d\vec{\mathbf{s}}$	Electric Potential $\Delta V = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$
Potential function, $V_G(\infty) = 0: V_G = -\dfrac{GM}{r}$	Potential function, $V(\infty) = 0: V = k_e\dfrac{Q}{r}$
$\lvert \Delta U_g \rvert = mgd$, (constant $\vec{\mathbf{g}}$)	$\lvert \Delta U \rvert = qEd$, (constant $\vec{\mathbf{E}}$)

4.3.1 Potential Energy in a System of Charges

Suppose you lift a mass m through a height h. The work done by the external agent (you), is positive, $W_{\text{ext}} = mgh > 0$. The work done by the gravitation field is negative, $W_g = -mgh = -W_{\text{ext}}$. The change in the potential energy is therefore equal to the work that you do in lifting the mass, $\Delta U_g = -W_g = +W_{\text{ext}} = mgh$.

If an electrostatic system of charges is assembled by an external agent, then $\Delta U = -W = +W_{\text{ext}}$. That is, the change in potential energy of the system is the work that must be put in by an external agent to assemble the configuration. The charges are brought in from infinity and are at rest at the end of the process. Let's start with just two charges q_1 and q_2 that are infinitely far apart with potential energy $U = 0$. Let the potential due to q_1 at a point P be V_1 (Figure 4.3.2).

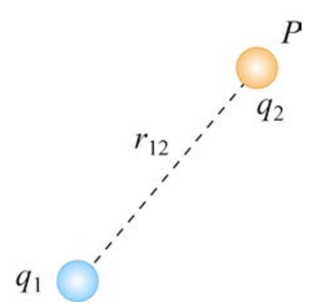


Figure 4.3.2 Two point charges separated by a distance r_{12}.

The work W_2 done by an external agent in bringing the second charge q_2 from infinity to P is then $W_2 = q_2 V_1$. Because $V_1 = q_1 / 4\pi\varepsilon_0 r_{12}$, where r_{12} is the distance measured from q_1 to P, we have that

$$U_{12} = W_2 = q_2 V_1 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} . \tag{4.3.5}$$

If q_1 and q_2 have the same sign, positive work must be done to overcome the electrostatic repulsion and the change in the potential energy of the system is positive, $U_{12} > 0$. On the other hand, if the signs are opposite, then $U_{12} < 0$ due to the attractive force between the charges. To add a third charge q_3 to the system (Figure 4.3.3), the work required is

$$W_3 = q_3 \left(V_1 + V_2 \right) = \frac{q_3}{4\pi\varepsilon_0} \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right). \tag{4.3.6}$$

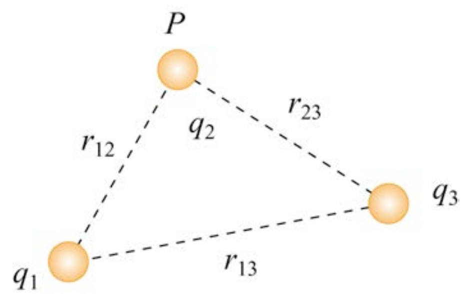


Figure 4.3.3 A system of three point charges.

The potential energy of this configuration is then

$$U = W_2 + W_3 = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) = U_{12} + U_{13} + U_{23} . \tag{4.3.7}$$

The equation shows that the total potential energy is simply the sum of the contributions from distinct pairs. Generalizing to a system of N charges, we have

$$U = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} \frac{q_i q_j}{r_{ij}} , \tag{4.3.8}$$

where the constraint $j > i$ is placed to avoid double counting each pair. Alternatively, one may count each pair twice and divide the result by 2. This leads to

$$U = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N} q_i \left(\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{r_{ij}} \right) = \frac{1}{2}\sum_{i=1}^{N} q_i V(r_i)\,. \qquad (4.3.9)$$

where $V(r_i)$, the quantity in the parenthesis is the potential at $\vec{\mathbf{r}}_i$ (location of q_i) due to all the other charges.

4.4 Deriving Electric Field from the Electric Potential

In Eq. (4.3.2) we established the relation between $\vec{\mathbf{E}}$ and *V*. If we consider two points that are separated by a small distance $d\vec{\mathbf{s}}$, the following differential form is obtained:

$$dV = -\vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}\,. \qquad (4.4.1)$$

In Cartesian coordinates, $\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$ and $d\vec{\mathbf{s}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$, and therefore

$$dV = (E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}\cdot)(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}) = E_x dx + E_y dy + E_z dz\,. \qquad (4.4.2)$$

We define *directional derivatives* $\partial V/\partial x$, $\partial V/\partial y$, and $\partial V/\partial z$ such that

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\,. \qquad (4.4.3)$$

Therefore

$$E_x = -\frac{\partial V}{\partial x},\quad E_y = -\frac{\partial V}{\partial y},\quad E_z = -\frac{\partial V}{\partial z}\,. \qquad (4.4.4)$$

By introducing a differential quantity called the *del (gradient) operator*

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \qquad (4.4.5)$$

the electric field can be written as

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}} \right) = -\nabla V. \qquad (4.4.6)$$

The differential operator, ∇, operates on a scalar quantity (electric potential) and results in a vector quantity (electric field). Mathematically, we can think of $\vec{\mathbf{E}}$ as the negative of the *gradient* of the electric potential V. Physically, the negative sign implies that if *V* increases as a positive charge moves along some direction, say *x*, with $\partial V/\partial x > 0$, then there is a non-vanishing component of $\vec{\mathbf{E}}$ in the opposite direction $E_x = -\partial V / \partial x < 0$. In the case of gravity, if the gravitational potential increases when a mass is lifted a distance *h*, the gravitational force must be downward.

If the charge distribution possesses spherical symmetry, then the resulting electric field is a function of the radial distance *r*, i.e., $\vec{\mathbf{E}} = E_r \hat{\mathbf{r}}$. In this case, $dV = -E_r\, dr$. If $V(r)$ is known, then $\vec{\mathbf{E}}$ may be obtained as

$$\vec{\mathbf{E}} = E_r \hat{\mathbf{r}} = -\left(\frac{dV}{dr}\right)\hat{\mathbf{r}} \tag{4.4.7}$$

For example, the electric potential due to a point charge q is $V(r) = q / 4\pi\varepsilon_0 r$. Using the above formula, the electric field is simply $\vec{\mathbf{E}} = (q / 4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$.

Example 4.4.1: Calculating Electric Field from Electric Potential

Suppose the electric potential due to a certain charge distribution can be written in Cartesian Coordinates as

$$V(x,y,z) = Ax^2y^2 + Bxyz$$

where A, B and C are constants. What is the associated electric field?

Solution: The electric field can be found by using Eq. (4.4.4)

$$\begin{aligned} E_x &= -\frac{\partial V}{\partial x} = -2Axy^2 - Byz \\ E_y &= -\frac{\partial V}{\partial y} = -2Ax^2y - Bxz \\ E_z &= -\frac{\partial V}{\partial z} = -Bxy \end{aligned}$$

Therefore, the electric field is $\vec{\mathbf{E}} = (-2Axy^2 - Byz)\hat{\mathbf{i}} - (2Ax^2y + Bxz)\hat{\mathbf{j}} - Bxy\hat{\mathbf{k}}$.

4.5 Gradients and Equipotentials

Suppose a system in two dimensions has an electric potential $V(x,y)$. The curves characterized by constant $V(x,y)$ are called *equipotential curves*. Examples of equipotential curves are depicted in Figure 4.5.1 below.

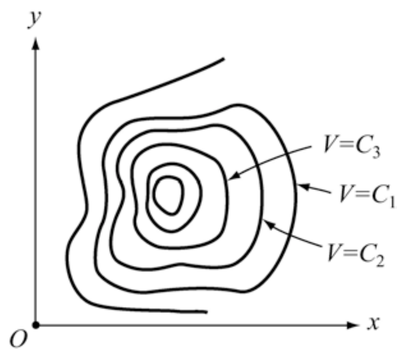


Figure 4.5.1 Equipotential curves

In three dimensions, surfaces such that $V(x,y,z) = \text{constant}$ are called *equipotential surfaces*. Because $\vec{\mathbf{E}} = -\nabla V$, we can show that the direction of $\vec{\mathbf{E}}$ at a point is always perpendicular to the equipotential through that point. We shall show this in two dimensions. Generalization to three dimensions is straightforward.

Referring to Figure 4.5.2, let the potential at a point $P(x,y)$ be $V(x,y)$. What is the potential difference dV between $P(x,y)$ and a neighboring point $P(x+dx, y+dy)$? Write the difference as

$$\begin{aligned} dV &= V(x+dx, y+dy) - V(x,y) \\ &= \left[V(x,y) + \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \cdots \right] - V(x,y) \approx \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \end{aligned} \tag{4.5.1}$$

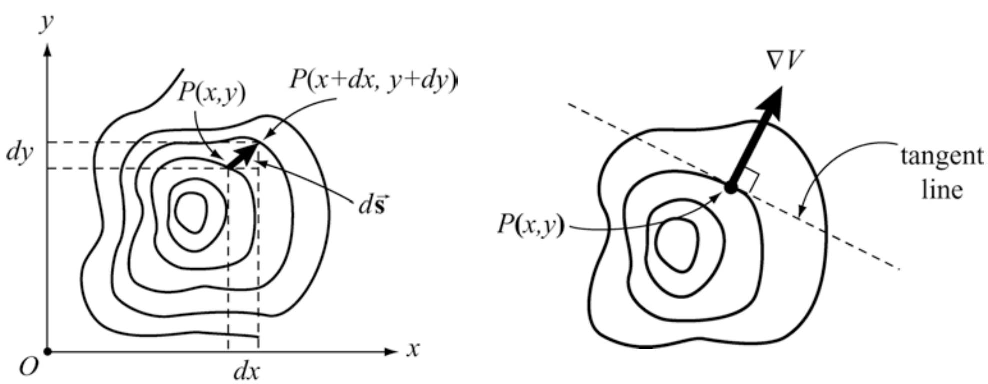


Figure 4.5.2 Change in *V* when moving from one equipotential curve to another

The displacement vector connecting the points is given by $d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$. We can rewrite dV as

$$dV = \left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} \right) \cdot \left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} \right) = (\nabla V) \cdot d\mathbf{s} = -\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \,. \tag{4.5.2}$$

If the displacement $d\vec{\mathbf{s}}$ is along the tangent to the equipotential curve that passes through the point P with coordinates (x, y), then $dV = 0$ because V is constant everywhere on that curve. This implies that $\vec{\mathbf{E}} \perp d\vec{\mathbf{s}}$ along the equipotential curve. That is, $\vec{\mathbf{E}}$ is perpendicular to the equipotential. In Figure 4.5.3 we illustrate some examples of equipotential curves. In three dimensions they become equipotential surfaces. From Eq. (4.5.8), we also see that the change in potential dV attains a maximum when the gradient ∇V is parallel to $d\vec{\mathbf{s}}$:

$$\max\left(\frac{dV}{ds}\right) = \left|\nabla V\right|. \tag{4.5.3}$$

Physically, this means that ∇V always points in the direction of maximum rate of change of V with respect to the displacement $d\vec{\mathbf{s}}$.

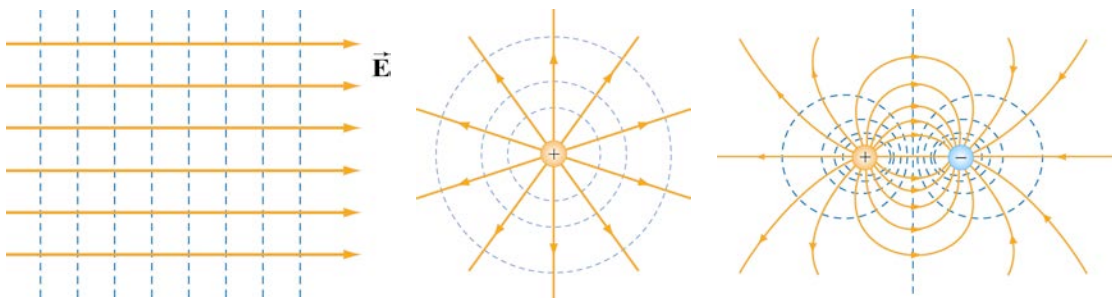


Figure 4.5.3 Equipotential curves and electric field lines for (a) a constant $\vec{\mathbf{E}}$ field, (b) a point charge, and (c) an electric dipole.

The properties of equipotential surfaces can be summarized as follows:

(i) The electric field lines are perpendicular to the equipotentials and point from higher to lower potentials.

(ii) By symmetry, the equipotential surfaces produced by a point charge form a family of concentric spheres, and for constant electric field, a family of planes perpendicular to the field lines.

(iii) The tangential component of the electric field along the equipotential surface is zero, otherwise non-vanishing work would be done to move a charge from one point on the surface to the other.

(iv) No work is required to move a particle along an equipotential surface.

A useful analogy for equipotential curves is a topographic map (Figure 4.5.4). Each contour line on the map represents a fixed elevation above sea level. Mathematically it is expressed as $z = f(x, y) = \text{constant}$. Since the gravitational potential near the surface of Earth is $V_g = gz$, these curves correspond to gravitational equipotentials.

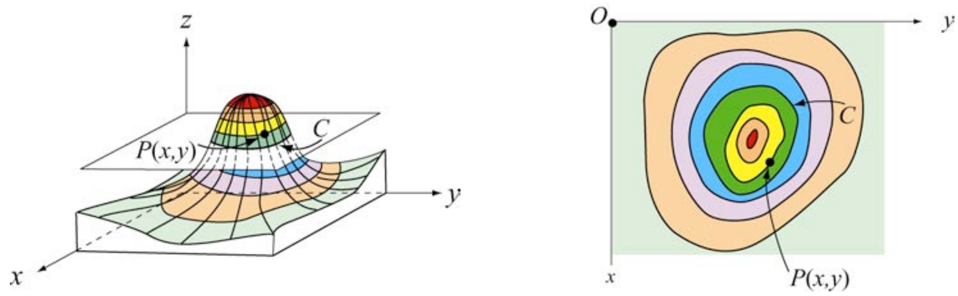


Figure 4.5.4 A topographic map

4.5.1: Conductors and Equipotentials

We already studies the basic properties of a conductor in Chapter 3 which we now summarized:

(1) the electric field inside a conductor is zero;

(2) any net charge must reside on the surface of the conductor;

(3) the tangential component of the electric field on the surface is zero;

(4) just outside the conductor, the electric field is normal to the surface;

(5) the discontinuity in the normal component of the electric field across the surface of a conductor is proportional to the surface charge density

Because the tangential component of the electric field on the surface of a conductor vanishes, this implies that the surface of a conductor in electrostatic equilibrium is an *equipotential surface*. To verify this claim, consider two points *A* and *B* on the surface of a conductor. Since the tangential component $E_t = 0$, the potential difference is

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$$

because $\vec{\mathbf{E}}$ is perpendicular to $d\vec{\mathbf{s}}$. Thus, points A and B are at the same potential with $V_A = V_B$.

4.6 Continuous Symmetric Charge Distributions

We shall now calculate the electric potential difference between two points in space associated with a continuous symmetric distribution of charge in which we can first use Gauss's Law to determine the electric field everywhere is space.

Example 4.61: Electric Potential Due to a Spherical Shell

Consider a metallic spherical shell of radius a and charge Q, as shown in Figure 4.6.1.

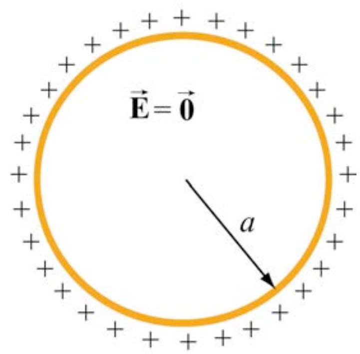


Figure 4.6.1 A spherical shell of radius a and charge Q.

(a) Find the electric potential everywhere.

(b) Calculate the potential energy of the system.

Solution:

(a) In Example 3.3, we showed that the electric field for a spherical shell of is given by

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > a \\ 0, & r < a. \end{cases}$$

The electric potential may be calculated by using Eq. (4.1.13),

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} .$$

For $r > a$, we have

$$V(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2} dr' = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} = k_e \frac{Q}{r}, \tag{4.6.1}$$

where we have chosen $V(\infty) = 0$ as our reference point. On the other hand, for $r < a,$ the potential becomes

$$\begin{aligned} V(r) - V(\infty) &= -\int_\infty^a E dr - \int_a^r 0 dr \\ &= -\int_\infty^a \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{1}{4\pi\varepsilon_0} \frac{Q}{a} = k_e \frac{Q}{a}. \end{aligned} \tag{4.6.2}$$

A plot of the electric potential is shown in Figure 4.6.2. Note that the potential V is constant inside a conductor.

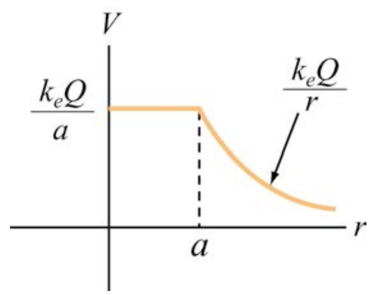


Figure 4.6.2 Electric potential as a function of r for a spherical conducting shell

(b) The potential energy U can be thought of as the work that needs to be done to build up the system. To charge up the sphere, an external agent must bring charge from infinity and deposit it onto the surface of the sphere.

Suppose the charge accumulated on the sphere at some instant is q. The potential at the surface of the sphere is then $V = q / 4\pi\varepsilon_0 a$. The amount of work that must be done by an external agent to bring charge dq from infinity and deposit it on the sphere is

$$dW_{\text{ext}} = V dq = \left(\frac{q}{4\pi\varepsilon_0 a} \right) dq. \tag{4.6.3}$$

Therefore, the total amount of work needed to charge the sphere to Q is

$$W_{\text{ext}} = \int_0^Q dq \frac{q}{4\pi\varepsilon_0 a} = \frac{Q^2}{8\pi\varepsilon_0 a}. \tag{4.6.4}$$

Because $V = Q / 4\pi\varepsilon_0 a$ and $W_{\text{ext}} = U$, the above expression simplified to

$$U = (1/2)QV. \tag{4.6.5}$$

The result can be contrasted with the case of a point charge. The work required to bring a point charge Q from infinity to a point where the electric potential due to other charges is V is $W_{\text{ext}} = QV$. Therefore, for a point charge Q, the potential energy is $U = QV$.

Example 4.6.2 Conducting Spheres Connected by a Wire

Why does lightning strike the tip of a lightning rod? Let's try to answer that question. Suppose two metal spheres with radii r_1 and r_2 are connected by a thin conducting wire, as shown in Figure 4.6.3.

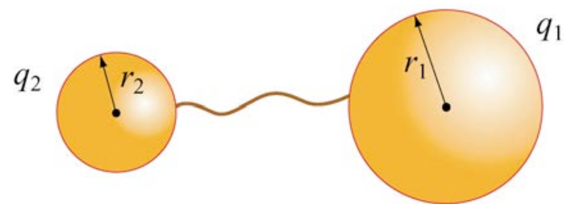


Figure 4.6.2 Two conducting spheres connected by a wire.

Charge will continue to flow until equilibrium is established such that both spheres are at the same potential $V_1 = V_2 = V$. Suppose the charges on the spheres at equilibrium are q_1 and q_2. Neglecting the effect of the wire that connects the two spheres, the equipotential condition implies

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2}.$$

Therefore

$$\frac{q_1}{r_1} = \frac{q_2}{r_2}, \tag{4.6.6}$$

provided that the two spheres are very far apart so that the charge distributions on the surfaces of the conductors are uniform. The electric fields can be expressed as

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^2} = \frac{\sigma_1}{\varepsilon_0}, \quad E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^2} = \frac{\sigma_2}{\varepsilon_0}, \tag{4.6.7}$$

where σ_1 and σ_2 are the surface charge densities on spheres 1 and 2, respectively. Divided the magnitudes of the electric fields yields

$$\frac{E_1}{E_2} = \frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1}. \tag{4.6.8}$$

With the surface charge density being inversely proportional to the radius, we conclude that the regions with the smallest radii of curvature have the greatest σ. Thus, the electric field strength on the surface of a conductor is greatest at the sharpest point. The design of a lightning rod is based on this principle. Lighting strikes the tip.

4.7 Continuous Non-Symmetric Charge Distributions

If the charge distribution is continuous, the potential at a point P can be found by summing over the contributions from individual differential elements of charge dq.

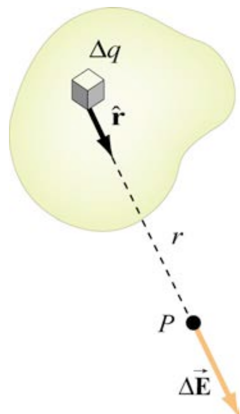


Figure 4.7.1 Continuous charge distribution

Consider the charge distribution shown in Figure 4.7.1. Taking infinity as our reference point with zero potential, the electric potential at P due to dq is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}. \tag{4.7.1}$$

Summing over contributions from all the differential elements, we have that

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}. \tag{4.7.2}$$

Example 4.7.1: Uniformly Charged Rod

Consider a non-conducting rod of length ℓ having a uniform charge density λ. Find the electric potential at P, a perpendicular distance y above the midpoint of the rod.

Solution: Consider a differential element of length dx' that carries a charge $dq = \lambda\, dx'$, as shown in Figure 4.7.2. The source element is located at $(x',0)$, while the field point P is located on the y-axis at $(0,y)$. The distance from dx' to P is $r = (x'^2 + y^2)^{1/2}$. Its contribution to the potential is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx'}{(x'^2+y^2)^{1/2}}.$$

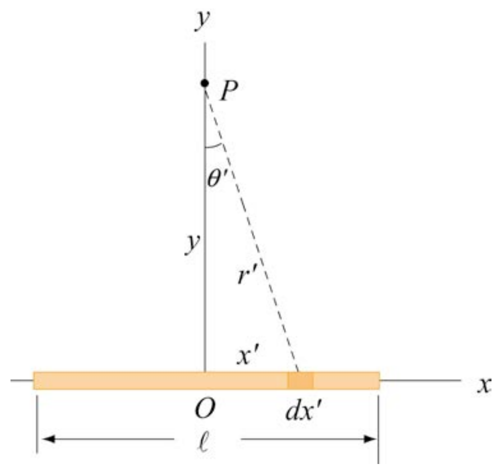


Figure 4.7.2 A non-conducting rod of length ℓ and uniform charge density λ.

Taking V to be zero at infinity, the total potential due to the entire rod is

$$\begin{aligned} V &= \frac{\lambda}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{dx'}{\sqrt{x'^2+y^2}} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[x'+\sqrt{x'^2+y^2}\right]\Bigg|_{-\ell/2}^{\ell/2} \\ &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2)+\sqrt{(\ell/2)^2+y^2}}{-(\ell/2)+\sqrt{(\ell/2)^2+y^2}}\right], \end{aligned} \tag{4.7.3}$$

where we have used the integration formula

$$\int\frac{dx'}{\sqrt{x'^2+y^2}} = \ln(x'+\sqrt{x'^2+y^2}).$$

A plot of $V(y)/V_0$, where $V_0 = \lambda/4\pi\varepsilon_0$, as a function of y/ℓ is shown in Figure 4.7.3.

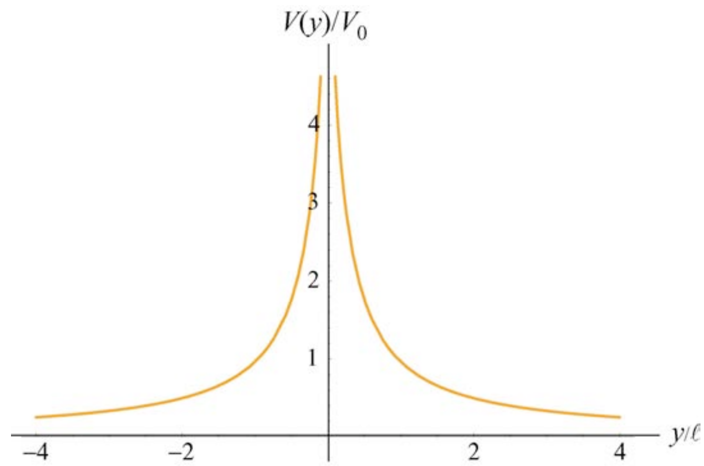


Figure 4.7.3 Electric potential along the axis that passes through the midpoint of a non-conducting rod.

In the limit $\ell \gg y$, the potential becomes

$$\begin{aligned} V &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}{-(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{1+\sqrt{1+(2y/\ell)^2}}{-1+\sqrt{1+(2y/\ell)^2}}\right] \\ &\approx \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{2}{2y^2/\ell^2}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\ell^2}{y^2}\right) \\ &= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\ell}{y}\right). \end{aligned} \tag{4.7.4}$$

The corresponding electric field can be obtained as

$$E_y = -\frac{\partial V}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0 y}\frac{\ell/2}{\sqrt{(\ell/2)^2+y^2}},$$

in agreement with the result obtained in Chapter 2, Eq. (2.10.9).

Example 4.7.2: Uniformly Charged Ring

Consider a uniformly charged ring of radius R and charge density λ (Figure 4.7.4). What is the electric potential at a distance z from the central axis?

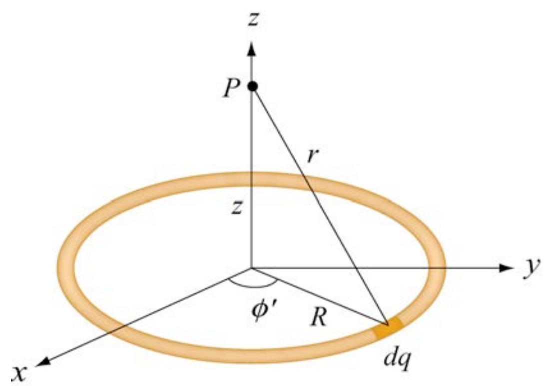


Figure 4.7.4 A non-conducting ring of radius R with uniform charge density λ.

Solution: Consider a small differential element $d\ell = R\,d\phi'$ on the ring. The element carries a charge $dq = \lambda\,d\ell = \lambda R\,d\phi'$, and its contribution to the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,d\phi'}{\sqrt{R^2+z^2}}.$$

The electric potential at P due to the entire ring is

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{\sqrt{R^2+z^2}}\oint d\phi' = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\lambda R}{\sqrt{R^2+z^2}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}, \qquad (4.7.5)$$

where we have substituted $Q = 2\pi R\lambda$ for the total charge on the ring. In the limit $z >> R$, the potential approaches its "point-charge" limit:

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{z}.$$

From Eq. (4.4.4) the z-component of the electric field may be obtained as

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}. \qquad (4.7.6)$$

in agreement with Eq. (2.10.14).

Example 4.7.3: Uniformly Charged Disk

Consider a uniformly charged disk of radius R and charge density σ lying in the xy-plane (Figure 4.7.5). What is the electric potential at a distance z from the central axis?

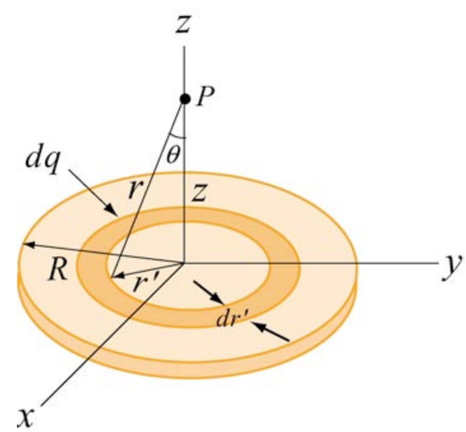


Figure 4.7.5 A non-conducting disk of radius R and uniform charge density σ.

Solution: Consider a circular ring of radius r' and width dr'. The charge on the ring is $dq' = \sigma dA' = \sigma(2\pi r' dr')$. The field point P is located along the z-axis a distance z from the plane of the disk. From the figure, we also see that the distance from a point on the ring to P is $r = (r'^2 + z^2)^{1/2}$. Therefore, the contribution to the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi r' dr')}{\sqrt{r'^2 + z^2}}.$$

By summing over all the rings that make up the disk, we have

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r' dr'}{\sqrt{r'^2 + z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r'^2 + z^2}\,\right]\Bigg|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2 + z^2} - |z|\,\right]. \qquad (4.7.7)$$

In the limit $|z| >> R$,

$$\sqrt{R^2 + z^2} = |z|\left(1 + \frac{R^2}{z^2}\right)^{1/2} = |z|\left(1 + \frac{R^2}{2z^2} + \cdots\right),$$

and the potential simplifies to the point-charge limit:

$$V \approx \frac{\sigma}{2\varepsilon_0}\cdot\frac{R^2}{2|z|} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(\pi R^2)}{|z|} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|z|}.$$

As expected, at large distance, the potential due to a non-conducting charged disk is the same as that of a point charge Q. A comparison of the electric potentials of the disk and a point charge is shown in Figure 4.7.6.

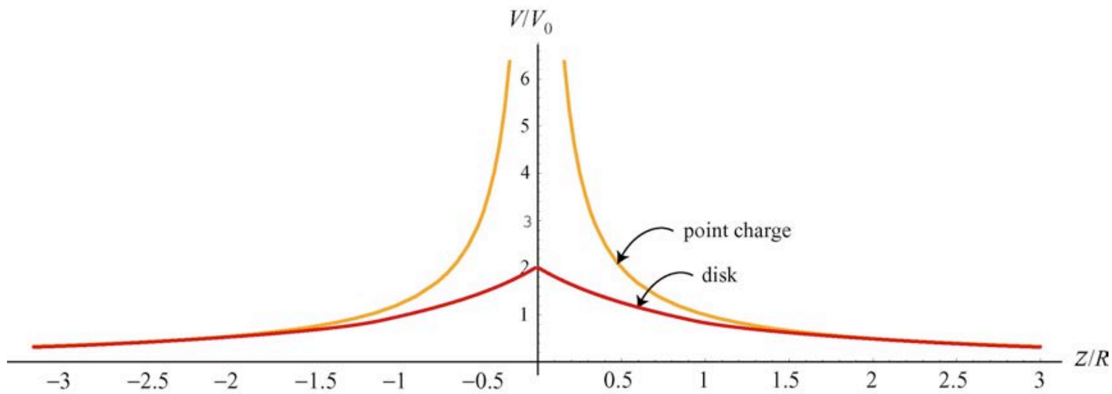


Figure 4.7.6 Comparison of the electric potentials of a non-conducting disk and a point charge. The electric potential is measured in terms of $V_0 = Q / 4\pi\varepsilon_0 R$.

Note that the electric potential at the center of the disk, $z = 0$, is finite, and its value is

$$V_c = \frac{\sigma R}{2\varepsilon_0} = \frac{Q}{\pi R^2} \cdot \frac{R}{2\varepsilon_0} = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{R} = 2V_0 . \tag{4.7.8}$$

This is the amount of work that needs to be done to bring a unit charge from infinity and place it at the center of the disk.

The corresponding electric field at P can be obtained as:

$$E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2 + z^2}}\right], \tag{4.7.9}$$

which agrees with Eq. (2.10.18). In the limit $R >> z$, the above equation becomes $E_z = \sigma / 2\varepsilon_0$, which is the electric field for an infinitely large non-conducting sheet.

4.8 Summary

- A force $\vec{\mathbf{F}}$ is **conservative** if the line integral of the force around a closed loop vanishes:

$$\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 0 .$$

- The change in potential energy associated with a conservative force $\vec{\mathbf{F}}$ acting on an object as it moves from A to B is

$$\Delta U = U_B - U_A = -\int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} .$$

- The **electric potential difference** ΔV between points A and B in an electric field $\vec{\mathbf{E}}$ is given by

$$\Delta V = V_B - V_A = \frac{\Delta U}{q_t} = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} .$$

  The quantity represents the amount of work done per unit charge to move a test charge q_t from point A to B, without changing its kinetic energy.

- The electric potential due to a point charge Q at a distance r away from the charge is

$$V = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} .$$

  For a collection of charges, using the superposition principle, the electric potential is

$$V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{Q_i}{r_i} .$$

- The **potential energy** associated with two point charges q_1 and q_2 separated by a distance r_{12} is

$$U = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} .$$

- From the electric potential V, the electric field may be obtained by taking the **gradient** of V,

$$\vec{\mathbf{E}} = -\nabla V .$$

  In Cartesian coordinates, the components may be written as

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}.$$

- The electric potential due to a continuous charge distribution is

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}.$$

4.9 Problem-Solving Strategy: Calculating Electric Potential

In this chapter, we showed how electric potential could be calculated for both the discrete and continuous charge distributions. Unlike electric field, electric potential is a scalar quantity. For the discrete distribution, we apply the superposition principle and sum over individual contributions:

$$V = k_e \sum_i \frac{q_i}{r_i}.$$

For the continuous distribution, we must evaluate the integral

$$V = k_e \int \frac{dq}{r}.$$

In analogy to the case of computing the electric field, we use the following steps to complete the integration:

(1) Start with $dV = k_e \frac{dq}{r}$.

(2) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda\, dl & \text{(length)} \\ \sigma\, dA & \text{(area)} \\ \rho\, dV & \text{(volume)} \end{cases}$$

depending on whether the charge is distributed over a length, an area, or a volume.

(3) Substitute dq into the expression for dV.

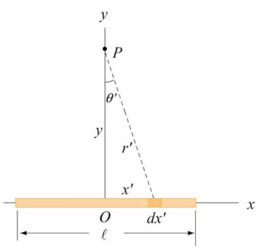
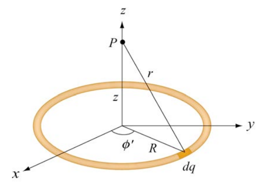
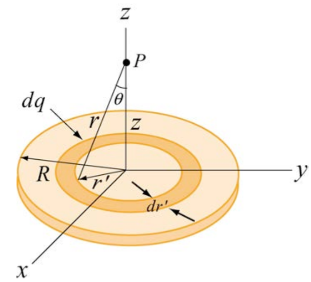
(4) Specify an appropriate coordinate system and express the differential element (dl, dA or dV) and r in terms of the coordinates (see Table 2.1.)

(5) Rewrite dV in terms of the integration variable.

(6) Complete the integration to obtain V.

Using the result obtained for V, one may calculate the electric field by $\vec{\mathbf{E}} = -\nabla V$. Furthermore, choosing a point P that lies sufficiently far away from the charge distribution can readily check the accuracy of the result. In this limit, if the charge distribution is of finite extent, the field should behave as if the distribution were a point charge, and falls off as $1/r^2$.

Below we illustrate how the above methodologies can be employed to compute the electric potential for a line of charge, a ring of charge and a uniformly charged disk.

	Charged Rod	Charged Ring	Charged disk
Figure			
(2) Express dq in terms of charge density	$dq = \lambda\, dx'$	$dq = \lambda\, dl$	$dq = \sigma\, dA$
(3) Substitute dq into expression for dV	$dV = k_e \dfrac{\lambda\, dx'}{r}$	$dV = k_e \dfrac{\lambda\, dl}{r}$	$dV = k_e \dfrac{\sigma\, dA}{r}$
(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $r = \sqrt{x'^2 + y^2}$	$dl = R\, d\phi'$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r'\, dr'$ $r = \sqrt{r'^2 + z^2}$
(5) Rewrite dV	$dV = k_e \dfrac{\lambda\, dx'}{(x'^2 + y^2)^{1/2}}$	$dV = k_e \dfrac{\lambda R\, d\phi'}{(R^2 + z^2)^{1/2}}$	$dV = k_e \dfrac{2\pi\sigma r'\, dr'}{(r'^2 + z^2)^{1/2}}$
(6) Integrate to get V	$V = \dfrac{\lambda}{4\pi\varepsilon_0} \displaystyle\int_{-\ell/2}^{\ell/2} \dfrac{dx'}{\sqrt{x'^2 + y^2}}$ $= \dfrac{\lambda}{4\pi\varepsilon_0} \ln\left[\dfrac{(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}{-(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}\right]$	$V = k_e \dfrac{R\lambda}{(R^2 + z^2)^{1/2}} \oint d\phi'$ $= k_e \dfrac{(2\pi R\lambda)}{\sqrt{R^2 + z^2}}$ $= k_e \dfrac{Q}{\sqrt{R^2 + z^2}}$	$V = k_e 2\pi\sigma \displaystyle\int_0^R \dfrac{r'\, dr'}{(r'^2 + z^2)^{1/2}}$ $= 2k_e\pi\sigma\left(\sqrt{z^2 + R^2} - \lvert z \rvert\right)$ $= \dfrac{2k_e Q}{R^2}\left(\sqrt{z^2 + R^2} - \lvert z \rvert\right)$
Derive E from V	$E_y = -\dfrac{\partial V}{\partial y}$ $= \dfrac{\lambda}{2\pi\varepsilon_0 y} \dfrac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = -\dfrac{\partial V}{\partial z} = \dfrac{k_e Q z}{(R^2 + z^2)^{3/2}}$	$E_z = -\dfrac{\partial V}{\partial z} = \dfrac{2k_e Q}{R^2}\left(\dfrac{z}{\lvert z \rvert} - \dfrac{z}{\sqrt{z^2 + R^2}}\right)$
Point-charge limit for E	$E_y \approx \dfrac{k_e Q}{y^2} \quad y \gg \ell$	$E_z \approx \dfrac{k_e Q}{z^2} \quad z \gg R$	$E_z \approx \dfrac{k_e Q}{z^2} \quad z \gg R$

4.10 Solved Problems

4.10.1 Electric Potential Due to a System of Two Charges

Consider a system of two charges shown in Figure 4.10.1.

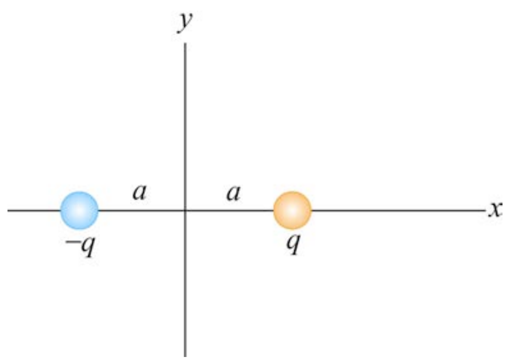


Figure 4.10.1 Electric dipole

Find the electric potential at an arbitrary point on the *x*-axis and make a plot.

Solution: The electric potential can be found by the superposition principle. At a point on the *x*-axis, we have

$$V(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x-a|} + \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|x+a|} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|}\right].$$

The above expression may be rewritten as

$$\frac{V(x)}{V_0} = \frac{1}{|x/a-1|} - \frac{1}{|x/a+1|},$$

where $V_0 = q/4\pi\varepsilon_0 a$.

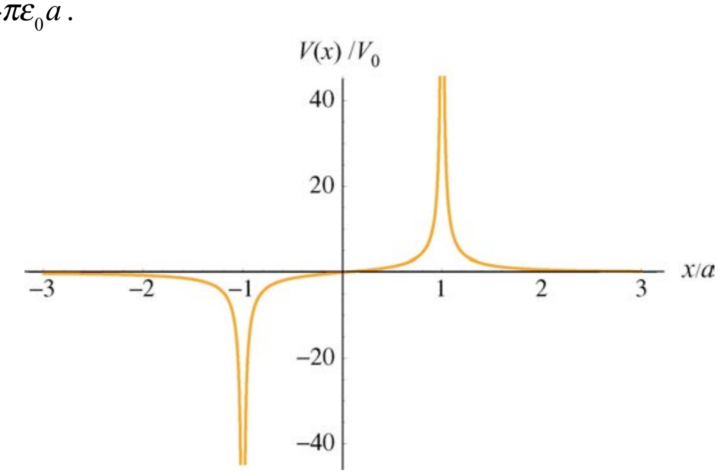


Figure 4.10.2

The plot of the dimensionless electric potential as a function of x/a. is depicted in Figure 4.10.2. As can be seen from the graph, $V(x)$ diverges at $x/a = \pm 1$, where the charges are located.

4.10.2 Electric Dipole Potential

Consider an electric dipole along the y-axis, as shown in the Figure 4.10.3. Find the electric potential V at a point P in the x-y plane, and use V to derive the corresponding electric field.

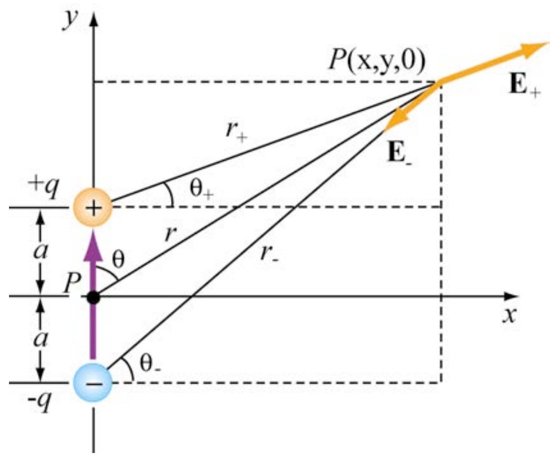


Figure 4.10.3

By superposition principle, the potential at P is given by

$$V = \sum_i V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right),$$

where $r_\pm^{\,2} = r^2 + a^2 \mp 2ra\cos\theta$. If we take the limit where $r >> a$, then

$$\frac{1}{r_\pm} = \frac{1}{r}\Big[1 + (a/r)^2 \mp 2(a/r)\cos\theta\Big]^{-1/2} = \frac{1}{r}\left[1 - \frac{1}{2}(a/r)^2 \pm (a/r)\cos\theta + \cdots\right].$$

The dipole potential can be approximated as

$$\begin{aligned} V &= \frac{q}{4\pi\varepsilon_0 r}\left[1 - \frac{1}{2}(a/r)^2 + (a/r)\cos\theta - 1 + \frac{1}{2}(a/r)^2 + (a/r)\cos\theta + \cdots\right] \\ &\approx \frac{q}{4\pi\varepsilon_0 r}\cdot\frac{2a\cos\theta}{r} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}, \end{aligned}$$

where $\vec{\mathbf{p}} = 2aq\,\hat{\mathbf{j}}$ is the electric dipole moment. In spherical polar coordinates, the gradient operator is

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Because the potential is now a function of both r and θ, the electric field will have components along the $\hat{\mathbf{r}}$ - and $\hat{\boldsymbol{\theta}}$ -directions. Using $\vec{\mathbf{E}} = -\nabla V$, we have

$$E_r = -\frac{\partial V}{\partial r} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}, \quad E_\phi = 0.$$

4.10.3 Electric Potential of an Annulus

Consider an annulus of uniform charge density σ, as shown in Figure 4.10.4. Find the electric potential at a point P along the symmetric axis.

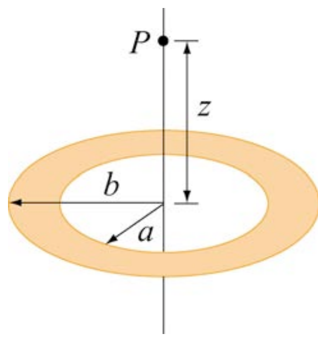


Figure 4.10.4 An annulus of uniform charge density.

Solution: Consider a small differential element dA at a distance r away from point P. The amount of charge contained in dA is given by

$$dq = \sigma dA = \sigma(r'd\theta)dr'.$$

Its contribution to the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma r'dr'd\theta}{\sqrt{r'^2 + z^2}}.$$

Integrating over the entire annulus, we obtain

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_a^b\int_0^{2\pi}\frac{r'dr'd\theta}{\sqrt{r'^2+z^2}} = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_a^b\frac{r'ds}{\sqrt{r'^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right],$$

where we have made used of the integral

$$\int \frac{ds\,s}{\sqrt{s^2+z^2}} = \sqrt{s^2+z^2}\ .$$

Notice that in the limit $a \to 0$ and $b \to R$, the potential becomes

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - |z|\right],$$

which agrees with the result of a non-conducting disk of radius R shown in Eq. (4.7.7).

4.10.4 Charge Moving Near a Charged Wire

A thin rod extends along the z-axis from $z=-d$ to $z=d$. The rod carries a positive charge Q uniformly distributed along its length $2d$ with charge density $\lambda = Q/2d$.

(a) Calculate the electric potential at a point $z > d$ along the z-axis.

(b) What is the change in potential energy if an electron moves from $z = 4d$ to $z = 3d$?

(c) If the electron started out at rest at the point $z = 4d$, what is its velocity at $z = 3d$?

Solutions:

(a) For simplicity, let's set the potential to be zero at infinity, $V(\infty) = 0$. Consider an infinitesimal charge element $dq = \lambda\, dz'$ located at a distance z' along the z-axis. Its contribution to the electric potential at a point $z > d$ is

$$dV = \frac{\lambda}{4\pi\varepsilon_0}\frac{dz'}{z-z'}.$$

Integrating over the entire length of the rod, we obtain

$$V(z) = \frac{\lambda}{4\pi\varepsilon_0}\int_{z+d}^{z-d}\frac{dz'}{z-z'} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{z+d}{z-d}\right).$$

(b) Using the result derived in (a), the electrical potential at $z = 4d$ is

$$V(z=4d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{4d+d}{4d-d}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{5}{3}\right).$$

Similarly, the electrical potential at $z = 3d$ is

$$V(z=3d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{3d+d}{3d-d}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln 2\,.$$

The electric potential difference between the two points is

$$\Delta V = V(z=3d) - V(z=4d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right) > 0\,.$$

Using the fact that the electric potential difference ΔV is equal to the change in potential energy per unit charge, we have

$$\Delta U = q\Delta V = -\frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right) < 0\,,$$

where $q = -e$ is the charge of the electron.

(c) If the electron starts out at rest at $z = 4d$ then the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_f^{\,2}\,.$$

By conservation of energy, the change in kinetic energy is

$$\Delta K = -\Delta U = \frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right) > 0\,.$$

Thus, the magnitude of the velocity at $z = 3d$ is

$$v_f = \sqrt{\frac{2|e|}{4\pi\varepsilon_0}\frac{\lambda}{m}\ln\left(\frac{6}{5}\right)}\,.$$

4.10.5 Electric Potential of a Uniformly Charged Sphere

An insulated solid sphere of radius a has a uniform charge density ρ. Compute the electric potential everywhere.

Solution: Using Gauss's law, we showed in Example 3.4 that the electric field due to the charge distribution is

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > a \\ \dfrac{Qr}{4\pi\varepsilon_0 a^3}\hat{\mathbf{r}}, & r < a. \end{cases} \tag{4.10.1}$$

The electric potential at P_1 (indicated in Figure 4.10.5) outside the sphere is

$$V_1(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = k_e\frac{Q}{r}. \tag{4.10.2}$$

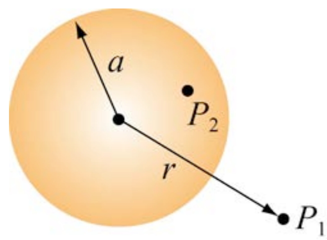


Figure 4.10.5

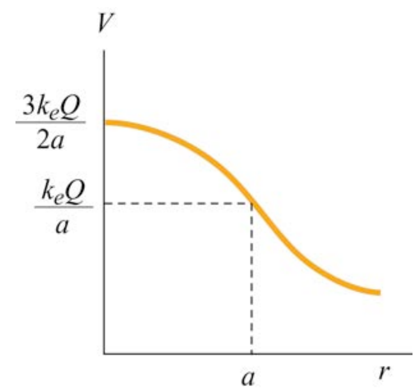


Figure 4.10.6 Electric potential due to a uniformly charged sphere as a function of r.

On the other hand, the electric potential at P_2 inside the sphere is given by

$$\begin{aligned} V_2(r) - V(\infty) &= -\int_\infty^a dr E(r > a) - \int_a^r E(r < a) = -\int_\infty^a dr \frac{Q}{4\pi\varepsilon_0 r^2} - \int_a^r dr' \frac{Qr}{4\pi\varepsilon_0 a^3} r' \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} - \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3}\frac{1}{2}\left(r^2 - a^2\right) = \frac{1}{8\pi\varepsilon_0}\frac{Q}{a}\left(3 - \frac{r^2}{a^2}\right) \\ &= k_e\frac{Q}{2a}\left(3 - \frac{r^2}{a^2}\right). \end{aligned} \tag{4.10.3}$$

A plot of electric potential as a function of r is given in Figure 4.10.6:

4.11 Conceptual Questions

1. What is the difference between electric potential and electric potential energy?

2. A uniform electric field is parallel to the x-axis. In what direction can a charge be displaced in this field without any external work being done on the charge?

3. Is it safe to stay in an automobile with a metal body during severe thunderstorm? Explain.

4. Why are equipotential surfaces always perpendicular to electric field lines?

5. The electric field inside a hollow, uniformly charged sphere is zero. Does this imply that the potential is zero inside the sphere?

4.12 Additional Problems

4.12.1 Cube

How much work is done to assemble eight identical point charges, each of magnitude q, at the corners of a cube of side a?

4.12.2 Three Charges

Three point-like objects with charges with $q = 3.00\times10^{-18}$ C and $q_1 = 6\times10^{-6}$ C are placed on the x-axis, as shown in the Figure 4.12.1. The distance between q and q_1 is a = 0.600 m.

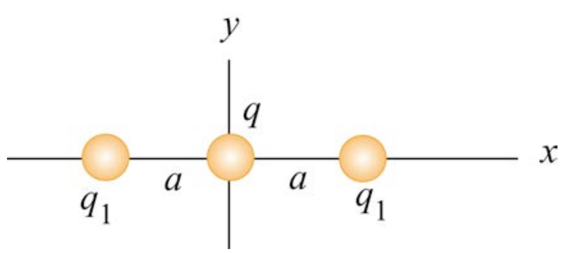


Figure 4.12.1

(a) What is the net force exerted on q by the other two charges q_1?

(b) What is the electric field at the origin due to the two charges q_1?

(c) What is the electric potential at the origin due to the two charges q_1?

4.12.3 Work Done on Charges

Two charges $q_1 = 3.0\,\mu\text{C}$ and $q_2 = -4.0\,\mu\text{C}$ initially are separated by a distance $r_0 = 2.0\,\text{cm}$. An external agent moves the charges until they are $r_f = 5.0\,\text{cm}$ apart.

(a) How much work is done by the electric field in moving the charges from r_0 to r_f? Is the work positive or negative?

(b) How much work is done by the external agent in moving the charges from r_0 to r_f? Is the work positive or negative?

(c) What is the potential energy of the initial state where the charges are $r_0 = 2.0\,\text{cm}$ apart?

(d) What is the potential energy of the final state where the charges are $r_f = 5.0\,\text{cm}$ apart?

(e) What is the change in potential energy from the initial state to the final state?

4.12.4 Calculating *E* from *V*

Suppose in some region of space the electric potential is given by

$$V(x,y,z) = V_0 - E_0 z + \frac{E_0 a^3 z}{(x^2 + y^2 + z^2)^{3/2}},$$

where a is a constant with dimensions of length. Find the *x, y,* and the *z*-components of the associated electric field.

4.12.5 Electric Potential of a Rod

A rod of length L lies along the x-axis with its left end at the origin and has a non-uniform charge density $\lambda = \alpha x$, where α is a positive constant (Figure 4.12.2).

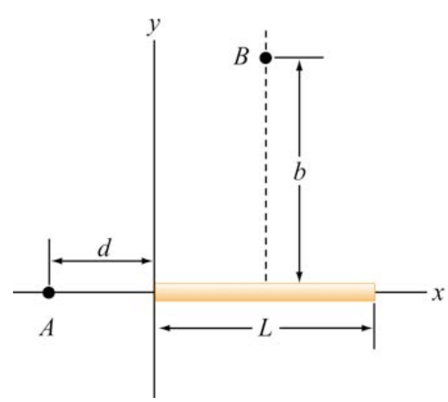


Figure 4.12.2

(a) What are the dimensions of α?

(b) Calculate the electric potential at A.

(c) Calculate the electric potential at point B that lies along the perpendicular bisector of the rod a distance b above the x-axis.

4.12.6 Electric Potential

Suppose that the electric potential in some region of space is given by

$$V(x,y,z) = V_0 \exp(-k\,|z|)\cos kx\,.$$

Find the electric field everywhere. Sketch the electric field lines in the xz -plane.

4.12.7 Calculating Electric Field from the Electric Potential

Suppose that the electric potential varies along the x-axis as shown in Figure 4.12.3 below.

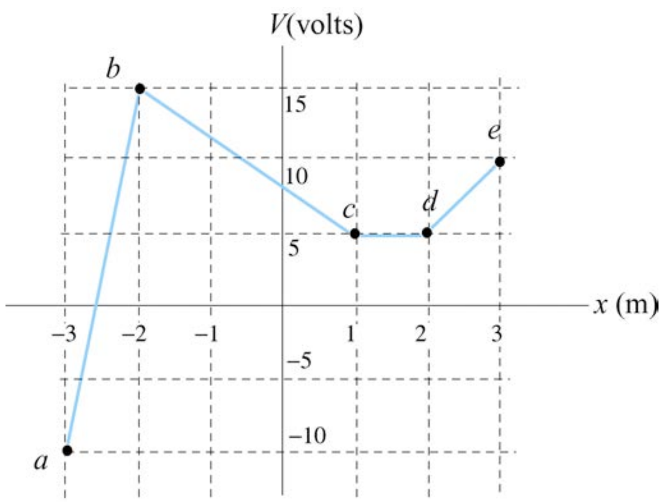


Figure 4.12.3

The potential does not vary in the y- or z-direction. Of the intervals shown (ignore the behavior at the end points of the intervals), determine the intervals in which E_x has

(a) its greatest absolute value. [Ans. 25 V/m in the interval ab.]

(b) its least absolute value. [Ans. (b) 0 V/m in the interval cd.]

(c) Plot E_x as a function of x.

(d) What sort of charge distributions would produce these kinds of changes in the potential? Where are they located? [Ans. sheets of charge extending in the yz-direction located at points b, c, d, etc. along the x-axis. Note again that a sheet of charge with charge per unit area σ will *always* produce a jump in the normal component of the electric field of magnitude σ / ε_0].

4.12.8 Electric Potential and Electric Potential Energy

A right isosceles triangle of side a has charges q, $+2q$, and $-q$ arranged on its vertices, as shown in Figure 4.12.4.

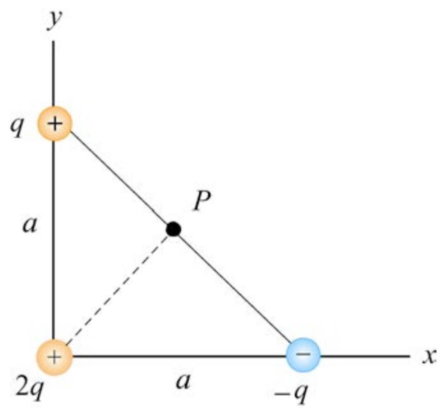


Figure 4.12.4

(a) What is the electric potential at point P, midway between the line connecting the $+q$ and $-q$ charges, assuming that $V = 0$ at infinity? [Ans. $q / \sqrt{2}\pi\varepsilon_0 a$.]

(b) What is the potential energy U of this configuration of three charges? What is the significance of the sign of your answer? [Ans. $-q^2 / 4\sqrt{2}\pi\varepsilon_0 a$, the negative sign means that work was done on the agent who assembled these charges in moving them in from infinity.]

(c) A fourth charge with charge $+3q$ is slowly moved in from infinity to point P. How much work must be done in this process? What is the significance of the sign of your answer? [Ans. $+3q^2 / \sqrt{2}\pi\varepsilon_0 a$, the positive sign means that work was done by the agent who moved this charge in from infinity.]

4.12.9. Electric Field, Potential and Energy

Three charges, $+5Q$, $-5Q$, and $+3Q$ are located on the y-axis at $y = +4a$, $y = 0$, and $y = -4a$, respectively. The point P is on the x-axis at $x = 3a$.

(a) How much energy did it take to assemble these charges?

(b) What are the x, y, and z components of the electric field $\vec{\mathbf{E}}$ at P?

(c) What is the electric potential V at point P, taking $V = 0$ at infinity?

(d) A fourth charge of $+Q$ is brought to P from infinity. What are the x, y, and z components of the force $\vec{\mathbf{F}}$ that is exerted on it by the other three charges?

(e) How much work was done (by the external agent) in moving the fourth charge $+Q$ from infinity to P?

4.12.10 P-N Junction

When two slabs of N-type and P-type semiconductors are put in contact, the relative affinities of the materials cause electrons to migrate out of the N-type material across the junction to the P-type material. This leaves behind a volume in the N-type material that is positively charged and creates a negatively charged volume in the P-type material.

Let us model this as two infinite slabs of charge, both of thickness a with the junction lying on the plane $z = 0$. The N-type material lies in the range $0 < z < a$ and has uniform charge density $+\rho_0$. The adjacent P-type material lies in the range $-a < z < 0$ and has uniform charge density $-\rho_0$. Thus:

$$\rho(x,y,z) = \rho(z) = \begin{cases} +\rho_0 & 0 < z < a \\ -\rho_0 & -a < z < 0 \\ 0 & |z| > a. \end{cases}$$

(a) Find the electric field everywhere.

(b) Find the potential difference between the points P_1 and $P_{2.}$. The point $P_{1.}$ is located on a plane parallel to the slab a distance $z_1 > a$ from the center of the slab. The point $P_{2.}$ is located on plane parallel to the slab a distance $z_2 < -a$ from the center of the slab.

4.12.11 Sphere with Non-Uniform Charge Distribution

A sphere made of insulating material of radius R has a charge density $\rho = ar$ where a is a constant. Let r be the distance from the center of the sphere.

(a) Find the electric field everywhere, both inside and outside the sphere.

(b) Find the electric potential everywhere, both inside and outside the sphere. Be sure to indicate where you have chosen your zero potential.

(c) How much energy does it take to assemble this configuration of charge?

(d) What is the electric potential difference between the center of the cylinder and a distance r inside the cylinder? Be sure to indicate where you have chosen your zero potential.

4.12.12 Electric Potential Energy of a Solid Sphere

Calculate the electric potential energy of a solid sphere of radius R filled with charge of uniform density ρ. Express your answer in terms of Q, the total charge on the sphere.

4.12.13 Calculating Electric Field from Electrical Potential

Figure 4.12.5 shows the variation of an electric potential V with distance z. The potential V does not depend on x or y. The potential V in the region $-1\,\text{m} < z < 1\,\text{m}$ is given in Volts by the expression $V(z) = 15 - 5z^2$. Outside of this region, the electric potential varies linearly with z, as indicated in the graph.

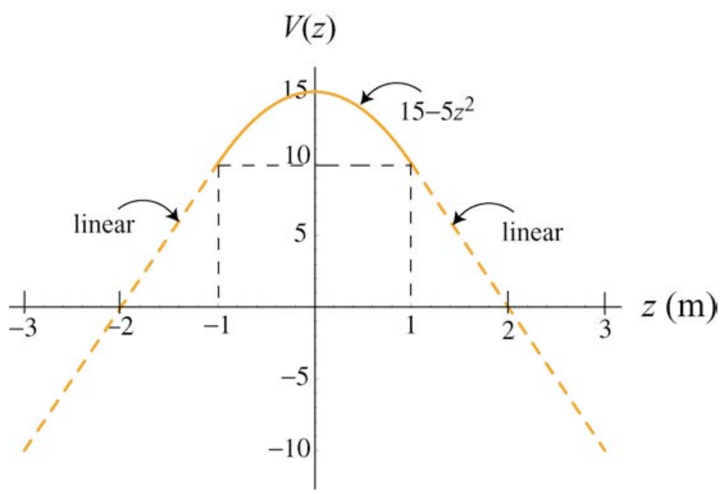


Figure 4.12.5

(a) Find an equation for the z-component of the electric field, E_z, in the region $-1\,\text{m} < z < 1\,\text{m}$.

(b) What is E_z in the region $z > 1$ m? Be careful to indicate the sign of E_z?

(c) What is E_z in the region $z < -1$ m? Be careful to indicate the sign of E_z?

(d) This potential is due a slab of charge with constant charge per unit volume ρ_0. Where is this slab of charge located (give the z-coordinates that bound the slab)? What is the charge density ρ_0 of the slab in C/m^3? Be sure to give clearly both the sign and magnitude of ρ_0.