Chapter 3

Gauss's Law

Gauss's Law

3.1 Electric Flux

In Chapter 2 we showed that the strength of an electric field is proportional to the number of field lines per area. The number of electric field lines that penetrates a given surface is called an "electric flux," which we denote as Φ_E. The electric field can therefore be thought of as the number of lines per unit area.

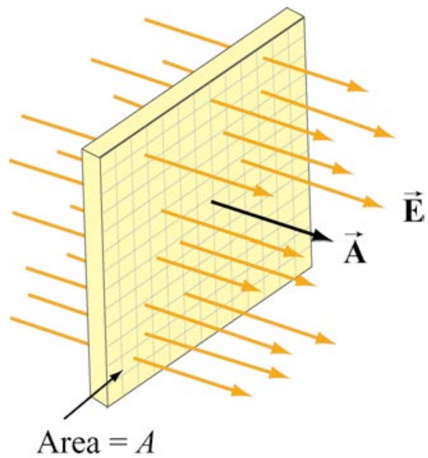


Figure 3.1.1 Electric field lines passing through a surface of area A.

Consider the surface shown in Figure 3.1.1. Let $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$ be defined as the *area vector* having a magnitude of the area of the surface, A, and pointing in the normal direction, $\hat{\mathbf{n}}$. If the surface is placed in a uniform electric field $\vec{\mathbf{E}}$ that points in the same direction as $\hat{\mathbf{n}}$, i.e., perpendicular to the surface A, the flux through the surface is

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, A = EA\,. \tag{3.1.1}$$

On the other hand, if the electric field $\vec{\mathbf{E}}$ makes an angle θ with $\hat{\mathbf{n}}$ (Figure 3.1.2), the electric flux becomes

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} = EA\cos\theta = E_{\mathrm{n}} A\,, \tag{3.1.2}$$

where $E_{\mathrm{n}} = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$ is the component of $\vec{\mathbf{E}}$ perpendicular to the surface.

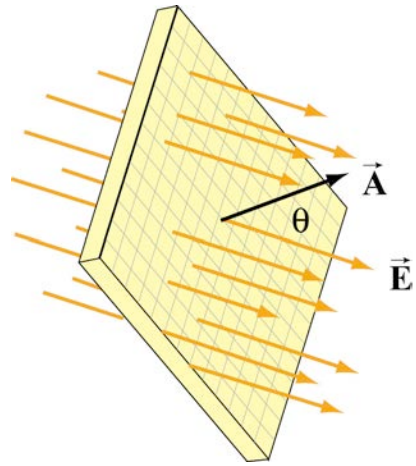


Figure 3.1.2 Electric field lines passing through a surface of area A whose normal makes an angle θ with the field.

Note that with the definition for the normal vector $\hat{\mathbf{n}}$, the electric flux Φ_E is positive if the electric field lines are leaving the surface, and negative if entering the surface.

In general, a surface S can be curved and the electric field $\vec{\mathbf{E}}$ may vary over the surface. We shall be interested in the case where the surface is *closed*. A closed surface is a surface that completely encloses a volume. In order to compute the electric flux, we divide the surface into a large number of infinitesimal area elements $\Delta\vec{\mathbf{A}}_i = \Delta A_i\,\hat{\mathbf{n}}_i$, as shown in Figure 3.1.3. Note that for a closed surface the unit vector $\hat{\mathbf{n}}_i$ is chosen to point in the *outward* normal direction.

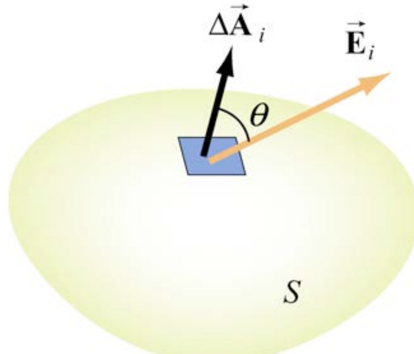


Figure 3.1.3 Electric field passing through an area element $\Delta\vec{\mathbf{A}}_i$, making an angle θ with the normal of the surface.

The electric flux through $\Delta\vec{\mathbf{A}}_i$ is

$$\Delta\Phi_E = \vec{\mathbf{E}}_i \cdot \Delta\vec{\mathbf{A}}_i = E_i \Delta A_i \cos\theta\,. \tag{3.1.3}$$

The total flux through the entire surface can be obtained by summing over all the area elements. Taking the limit $\Delta\vec{\mathbf{A}}_i \to 0$ and the number of elements to infinity, we have

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum \vec{\mathbf{E}}_i \cdot d\vec{\mathbf{A}}_i = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}\,, \tag{3.1.4}$$

where the symbol $\oiint_S$ denotes a double integral over a *closed* surface *S*. In order to evaluate the above integral, we must first specify the surface and then sum over the dot product $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$.

3.2 Gauss's Law (see also Gauss's Law Simulation in Section 3.10)

We now introduce Gauss's Law. Many of the conceptual problems students have with Gauss's Law have to do with understanding the geometry, and we urge you to read the standard development below and then go to the Gauss's Law simulation in Section 3.10. There you can interact directly with the relevant geometry in a 3D interactive simulation of Gauss's Law.

Consider a positive point charge Q located at the center of a sphere of radius r, as shown in Figure 3.2.1. The electric field due to the charge Q is $\vec{\mathbf{E}} = (Q / 4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$, which points in the radial direction. We enclose the charge by an imaginary sphere of radius r called the "Gaussian surface."

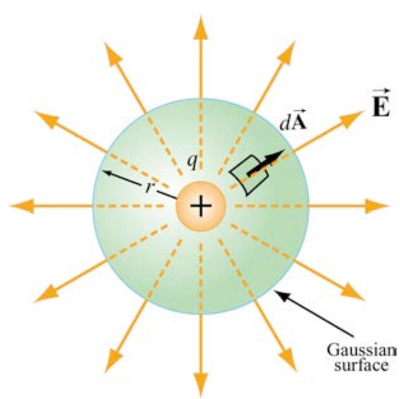


Figure 3.2.1 A spherical Gaussian surface enclosing a charge Q.

In spherical coordinates, a small surface area element on the sphere is given by (Figure 3.2.2)

$$d\vec{\mathbf{A}} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}\,. \tag{3.2.1}$$

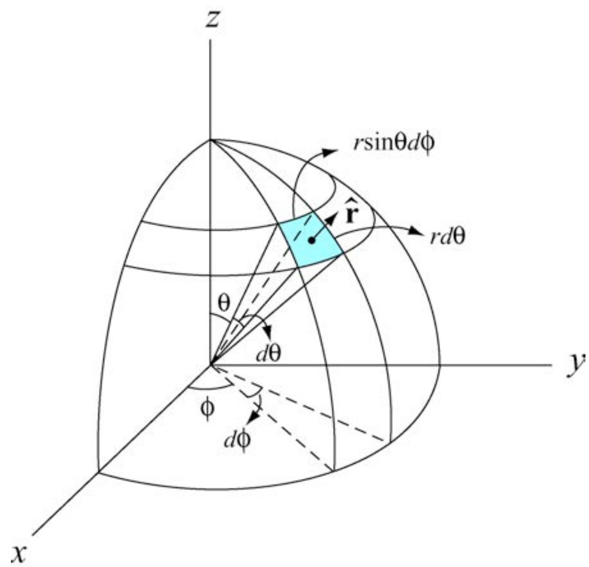


Figure 3.2.2 A small area element on the surface of a sphere of radius *r*.

Thus, the net electric flux through the area element is

$$d\Phi_E = \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\,dA = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\right)\left(r^2 \sin\theta\, d\theta\; d\phi\right) = \frac{Q}{4\pi\varepsilon_0}\sin\theta\, d\theta\; d\phi\,. \qquad (3.2.2)$$

The total flux through the entire surface is

$$\Phi_E = \oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{4\pi\varepsilon_0}\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{Q}{\varepsilon_0}. \qquad (3.2.3)$$

The same result can also be obtained by noting that a sphere of radius *r* has a surface area $A = 4\pi r^2$, and since the magnitude of the electric field at any point on the spherical surface is $E = Q / 4\pi\varepsilon_0 r^2$, the electric flux through the surface is

$$\Phi_E = \oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\oint\!\!\oint_S dA = EA = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\right)4\pi r^2 = \frac{Q}{\varepsilon_0}. \qquad (3.2.4)$$

In the above, we have chosen a sphere to be the Gaussian surface. However, it turns out that the shape of the closed surface can be arbitrarily chosen. For the surfaces shown in Figure 3.2.3, the same result ($\Phi_E = Q / \varepsilon_0$) is obtained, whether the choice is S_1, S_2 or S_3.

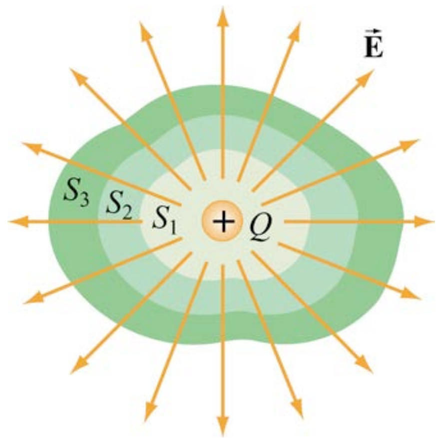


Figure 3.2.3 Different Gaussian surfaces with the same outward electric flux.

The statement that the net flux through any closed surface is proportional to the net charge enclosed is known as Gauss's law. Mathematically, Gauss's law is expressed as

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0} \qquad \text{(Gauss's Law)}, \tag{3.2.5}$$

where q_{enc} is the net charge inside the surface. One way to explain why Gauss's law holds is due to note that the number of field lines that leave the charge is independent of the shape of the imaginary Gaussian surface we choose to enclose the charge.

To prove Gauss's law, we introduce the concept of the *solid angle*. Let $\Delta\vec{\mathbf{A}}_1 = \Delta A_1 \hat{\mathbf{r}}$ be an area element on the surface of a sphere S_1 of radius r_1, as shown in Figure 3.2.4.

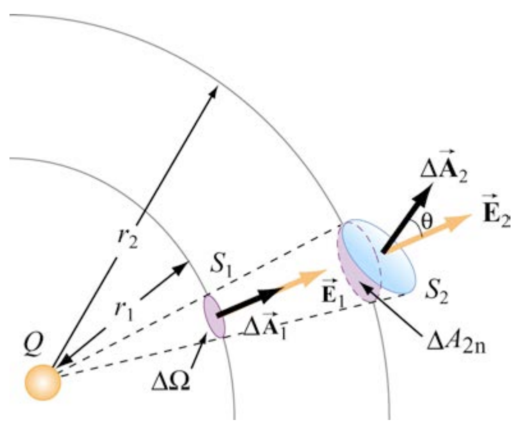


Figure 3.2.4 The area element ΔA subtends a solid angle $\Delta\Omega$.

The solid angle $\Delta\Omega$ subtended by $\Delta\vec{\mathbf{A}}_1 = \Delta A_1 \hat{\mathbf{r}}$ at the center of the sphere is defined as

$$\Delta\Omega \equiv \frac{\Delta A_1}{r_1^2}. \tag{3.2.6}$$

Solid angles are dimensionless quantities measured in steradians [sr]. Since the surface area of the sphere S_1 is $4\pi r_1^2$, the total solid angle subtended by the sphere is

$$\Omega = \frac{4\pi r_1^2}{r_1^2} = 4\pi. \tag{3.2.7}$$

The concept of solid angle in three dimensions is analogous to the ordinary angle in two dimensions.

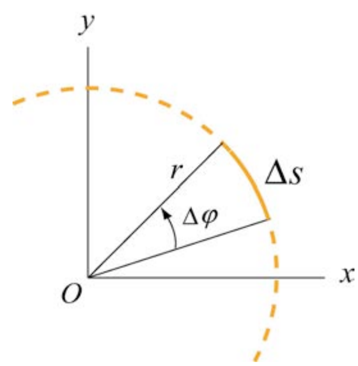


Figure 3.2.5 The arc Δs subtends an angle $\Delta\varphi$.

As illustrated in Figure 3.2.5, an angle $\Delta\varphi$ is the ratio of the length of the arc to the radius r of a circle:

$$\Delta\varphi = \frac{\Delta s}{r}. \tag{3.2.8}$$

Because the total length of the arc is $s = 2\pi r$, the total angle subtended by the circle is

$$\varphi = \frac{2\pi r}{r} = 2\pi. \tag{3.2.9}$$

In Figure 3.2.4, the area element $\Delta\vec{\mathbf{A}}_2$ makes an angle θ with the radial unit vector $\hat{\mathbf{r}}$, therefore the solid angle subtended by ΔA_2 is

$$\Delta\Omega = \frac{\Delta\vec{\mathbf{A}}_2 \cdot \hat{\mathbf{r}}}{r_2^2} = \frac{\Delta A_2 \cos\theta}{r_2^2} = \frac{\Delta A_{2\mathrm{n}}}{r_2^2}, \tag{3.2.10}$$

where $\Delta A_{2n} = \Delta A_2 \cos\theta$ is the area of the radial projection of ΔA_2 onto a second sphere S_2 of radius r_2, concentric with S_1. As shown in Figure 3.2.4, the solid angle subtended is the same for both ΔA_1 and ΔA_{2n}:

$$\Delta\Omega = \frac{\Delta A_1}{r_1^2} = \frac{\Delta A_2 \cos\theta}{r_2^2}. \tag{3.2.11}$$

Now suppose a point charge Q is placed at the center of the concentric spheres. The electric field strengths E_1 and E_2 at the center of the area elements ΔA_1 and ΔA_2 are related by Coulomb's law:

$$E_i = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_i^2} \quad \Rightarrow \quad \frac{E_2}{E_1} = \frac{r_1^2}{r_2^2}. \tag{3.2.12}$$

The electric flux through ΔA_1 on S_1 is

$$\Delta\Phi_1 = \vec{\mathbf{E}} \cdot \Delta\vec{\mathbf{A}}_1 = E_1 \Delta A_1. \tag{3.2.13}$$

On the other hand, the electric flux through ΔA_2 on S_2 is $\Delta A_2 \cos\theta$

$$\Delta\Phi_2 = \vec{\mathbf{E}}_2 \cdot \Delta\vec{\mathbf{A}}_2 = E_2 \,\Delta A_2 \cos\theta = E_1\left(\frac{r_1^2}{r_2^2}\right)\cdot\left(\frac{r_2^2}{r_1^2}\right)A_1 = E_1 \,\Delta A_1 = -\vec{\mathbf{E}}_1 \cdot \Delta\vec{\mathbf{A}}_1 = -\Delta\Phi_1. \tag{3.2.14}$$

Thus, we see that the electric flux through any area element subtending the same solid angle is constant, independent of the shape or orientation of the surface.

In summary, Gauss's law provides a convenient tool for evaluating electric field. However, its application is limited only to systems that possess certain symmetry, namely, systems with cylindrical, planar and spherical symmetry. In the table below, we give some examples of systems in which Gauss's law is applicable for determining electric field, with the corresponding Gaussian surfaces:

Symmetry	**System**	**Gaussian Surface**	**Examples**
Cylindrical	Infinite rod	Coaxial Cylinder	Example 3.1
Planar	Infinite plane	Gaussian "Pillbox"	Example 3.2
Spherical	Sphere, Spherical shell	Concentric Sphere	Examples 3.3 & 3.4

The following steps may be useful when applying Gauss's law:

(1) Identify the symmetry associated with the charge distribution.

(2) Determine the direction of the electric field, and a "Gaussian surface" on which the magnitude of the electric field is constant over portions of the surface.

(3) Divide the space into different regions associated with the charge distribution. For each region, calculate q_{enc}, the charge enclosed by the Gaussian surface.

(4) Calculate the electric flux Φ_E through the Gaussian surface for each region.

(5) Equate Φ_E with $q_{\text{enc}} / \varepsilon_0$, and deduce the magnitude of the electric field.

Example 3.1: Infinitely Long Rod of Uniform Charge Density

An infinitely long rod of negligible radius has a uniform charge density λ. Calculate the electric field at a distance r from the wire.

Solution: We shall solve the problem by following the steps outlined above.

(1) An infinitely long rod possesses cylindrical symmetry.

(2) The charge density is uniformly distributed throughout the length, and the electric field $\vec{\mathbf{E}}$ must be point radially away from the symmetry axis of the rod (Figure 3.2.6). The magnitude of the electric field is constant on cylindrical surfaces of radius r. Therefore, we choose a coaxial cylinder as our Gaussian surface.

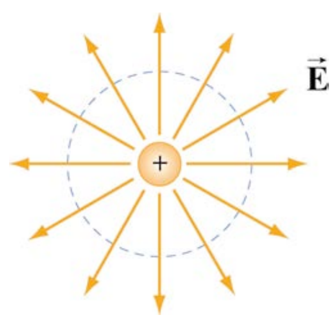


Figure 3.2.6 Field lines for an infinite uniformly charged rod (the symmetry axis of the rod and the Gaussian cylinder are perpendicular to plane of the page.)

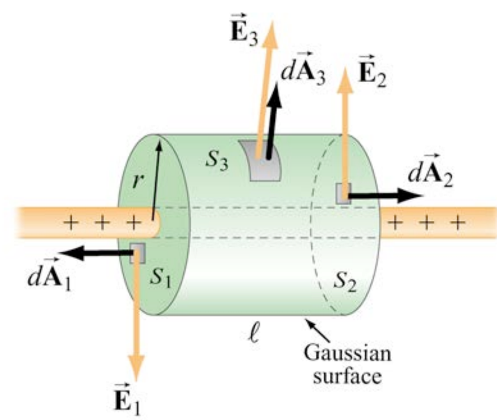


Figure 3.2.7 Gaussian surface for a uniformly charged rod.

(3) The amount of charge enclosed by the Gaussian surface, a cylinder of radius r and length ℓ (Figure 3.2.7), is $q_{\text{enc}} = \lambda\ell$.

(4) As indicated in Figure 3.2.7, the Gaussian surface consists of three parts: two end-cap surfaces S_1 and S_2 plus the cylindrical sidewall S_3. The flux through the Gaussian surface is

$$\begin{aligned}\Phi_E &= \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 \\ &= 0 + 0 + E_3 A_3 = E(2\pi r \ell),\end{aligned} \tag{3.2.15}$$

where we have set $E_3 = E$. As can be seen from the figure, no flux passes through the ends since the area vectors $d\vec{\mathbf{A}}_1$ and $d\vec{\mathbf{A}}_2$ are perpendicular to the electric field which points in the radial direction.

(5) Applying Gauss's law gives $E(2\pi r\ell) = \lambda\ell / \varepsilon_0$, or

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}. \tag{3.2.16}$$

The result is in complete agreement with that obtained in Eq. (2.10.11) using Coulomb's law. Notice that the result is independent of the length ℓ of the cylinder, and only depends on the inverse of the distance r from the symmetry axis. The qualitative behavior of E as a function of r is plotted in Figure 3.2.8.

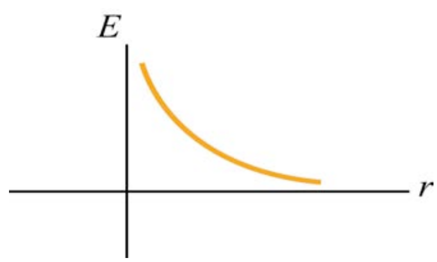


Figure 3.2.8 Electric field due to a uniformly charged rod as a function of r.

Example 3.2: Infinite Plane of Charge

Consider an infinitely large non-conducting plane in the *xy*-plane with uniform surface charge density σ. Determine the electric field everywhere in space.

Solution:

(1) An infinitely large plane possesses a planar symmetry.

(2) Since the charge is uniformly distributed on the surface, the electric field $\vec{\mathbf{E}}$ must point perpendicularly away from the plane, $\vec{\mathbf{E}} = E\,\hat{\mathbf{k}}$. The magnitude of the electric field is constant on planes parallel to the non-conducting plane.

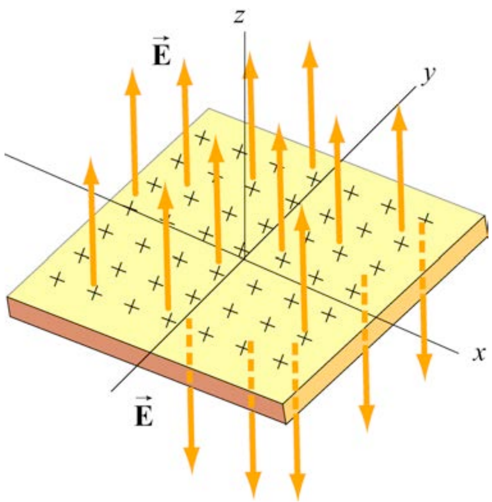


Figure 3.2.9 Electric field for uniform plane of charge

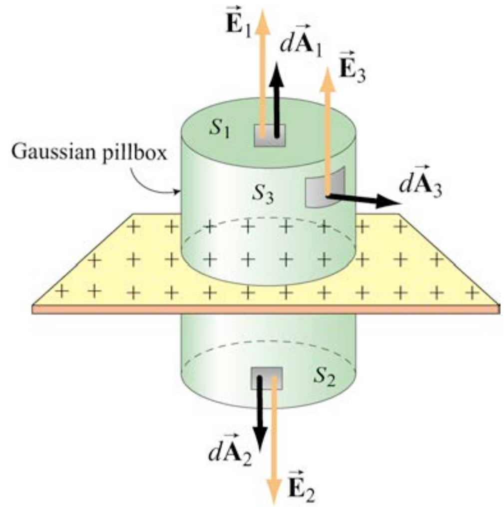


Figure 3.2.10 A Gaussian "pillbox" for calculating the electric field due to a large plane.

We choose our Gaussian surface to be a cylinder, which is often referred to as a "pillbox" (Figure 3.2.10). The pillbox also consists of three parts: two end-caps S_1 and S_2, and a curved side S_3.

(3) Since the surface charge distribution on is uniform, the charge enclosed by the Gaussian "pillbox" is $q_{\text{enc}} = \sigma A$, where $A = A_1 = A_2$ is the area of the end-caps.

(4) The total flux through the Gaussian pillbox flux is

$$\begin{aligned}\Phi_E &= \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 \\ &= E_1 A_1 + E_2 A_2 + 0 \\ &= (E_1 + E_2) A.\end{aligned} \tag{3.2.17}$$

Because the two ends are at the same distance from the plane, by symmetry, the magnitude of the electric field must be the same: $E_1 = E_2 = E$. Hence, the total flux can be rewritten as

$$\Phi_E = 2EA. \tag{3.2.18}$$

(5) By applying Gauss's law, we obtain

$$2EA = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}.$$

Therefore the magnitude of the electric field is

$$E = \frac{\sigma}{2\varepsilon_0}. \tag{3.2.19}$$

In vector notation, we have

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < 0. \end{cases} \tag{3.2.20}$$

Thus, we see that the electric field due to an infinite large non-conducting plane is uniform in space. The result, plotted in Figure 3.2.11, is the same as that obtained in Eq. (2.10.20) using Coulomb's law.

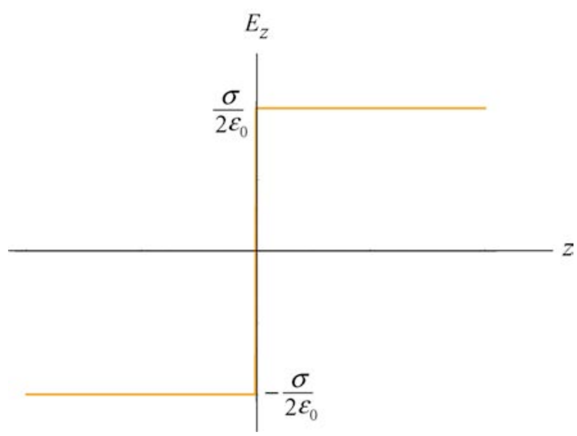


Figure 3.2.11 Electric field of an infinitely large non-conducting plane.

Note again the discontinuity in electric field as we cross the plane:

$$\Delta E_z = E_{z+} - E_{z-} = \frac{\sigma}{2\varepsilon_0} - \left(-\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0}. \tag{3.2.21}$$

Example 3.3: Spherical Shell

A thin spherical shell of radius a has a charge $+Q$ evenly distributed over its surface. Find the electric field both inside and outside the shell.

Solutions:

The charge distribution is spherically symmetric, with a surface charge density $\sigma = Q / A_s = Q / 4\pi a^2$, where $A_s = 4\pi a^2$ is the surface area of the sphere. The electric field $\vec{\mathbf{E}}$ must be radially symmetric and directed outward (Figure 3.2.12). We treat the regions $r \le a$ and $r \ge a$ separately.

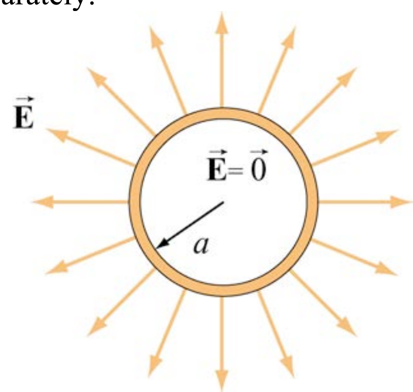


Figure 3.2.12 Electric field for uniform spherical shell of charge

Case 1: $r \le a$. We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 3.2.13(a).

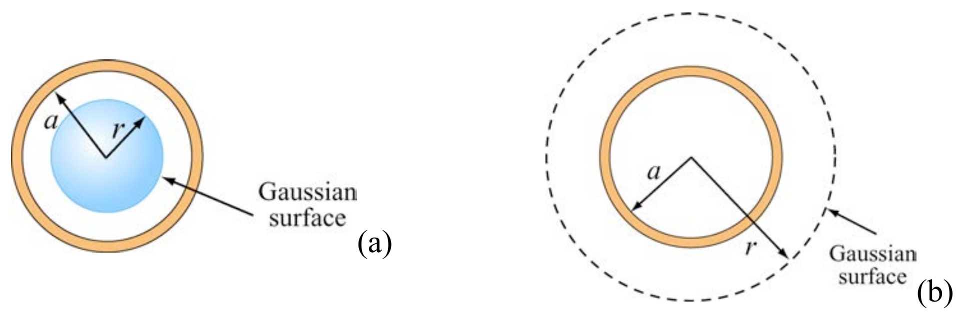


Figure 3.2.13 Gaussian surface for uniformly charged spherical shell for (a) $r < a$, and (b) $r \ge a$.

The charge enclosed by the Gaussian surface is $q_{\text{enc}} = 0$ since all the charge is located on the surface of the shell. Thus, from Gauss's law, $\Phi_E = q_{\text{enc}} / \varepsilon_0$, we conclude

$$E = 0, \quad r < a. \tag{3.2.22}$$

Case 2: $r \ge a$. In this case, the Gaussian surface is a sphere of radius $r \ge a$, as shown in Figure 3.2.13(b). Since the radius of the "Gaussian sphere" is greater than the radius of the spherical shell, all the charge is enclosed $q_{\text{enc}} = Q$. Because the flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2),$$

by applying Gauss's law, we obtain

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = k_e \frac{Q}{r^2}, \quad r \ge a. \tag{3.2.23}$$

Note that the field outside the sphere is the same as if all the charges were concentrated at the center of the sphere. The qualitative behavior of E as a function of r is plotted in Figure 3.2.14.

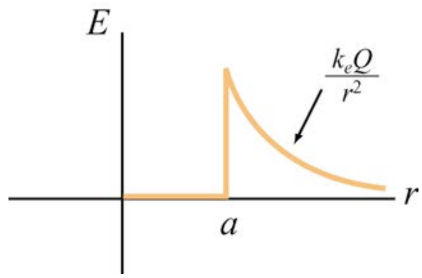


Figure 3.2.14 Electric field as a function of r due to a uniformly charged spherical shell.

As in the case of a non-conducting charged plane, we again see a discontinuity in E as we cross the boundary at $r = a$. The change, from outer to the inner surface, is given by

$$\Delta E = E_+ - E_- = \frac{Q}{4\pi\varepsilon_0 a^2} - 0 = \frac{\sigma}{\varepsilon_0}.$$

Example 3.4: Non-Conducting Solid Sphere

An electric charge $+Q$ is uniformly distributed throughout a non-conducting solid sphere of radius a. Determine the electric field everywhere inside and outside the sphere.

Solution: The charge distribution is spherically symmetric with the charge density given by

$$\rho = \frac{Q}{V} = \frac{Q}{(4/3)\pi a^3}, \tag{3.2.24}$$

where V is the volume of the sphere. In this case, the electric field $\vec{\mathbf{E}}$ is radially symmetric and directed outward. The magnitude of the electric field is constant on spherical surfaces of radius r. The regions $r \le a$ and $r \ge a$ shall be studied separately.

Case 1: $r \le a$. We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 3.2.15(a).

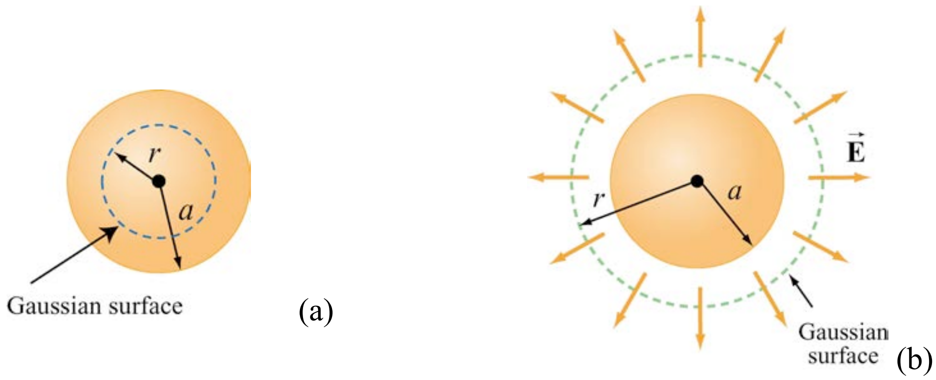


Figure 3.2.15 Gaussian surface for uniformly charged solid sphere, for (a) $r \le a$, and (b) $r > a$.

The flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2) .$$

With uniform charge distribution, the charge enclosed is

$$q_{\text{enc}} = \int_V \rho \, dV = \rho V = \rho\left(\frac{4}{3}\pi r^3\right) = Q\left(\frac{r^3}{a^3}\right), \tag{3.2.25}$$

which is proportional to the volume enclosed by the Gaussian surface. Applying Gauss's law $\Phi_E = q_{\text{enc}} / \varepsilon_0$, we obtain

$$E(4\pi r^2) = \frac{\rho}{\varepsilon_0}\left(\frac{4}{3}\pi r^3\right)$$

The magnitude of the electric field is therefore

$$E = \frac{\rho r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 a^3}, \quad r \le a. \tag{3.2.26}$$

Case 2: $r \ge a$. In this case, our Gaussian surface is a sphere of radius $r \ge a$, as shown in Figure 3.2.15(b). Since the radius of the Gaussian surface is greater than the radius of the sphere all the charge is enclosed in our Gaussian surface: $q_{\text{enc}} = Q$. With the electric flux through the Gaussian surface given by $\Phi_E = E(4\pi r^2)$, upon applying Gauss's law, we obtain $E(4\pi r^2) = Q / \varepsilon_0$. The magnitude of the electric field is therefore

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = k_e \frac{Q}{r^2}, \quad r > a. \tag{3.2.27}$$

The field outside the sphere is the same as if all the charges were concentrated at the center of the sphere. The qualitative behavior of E as a function of r is plotted in Figure 3.2.16.

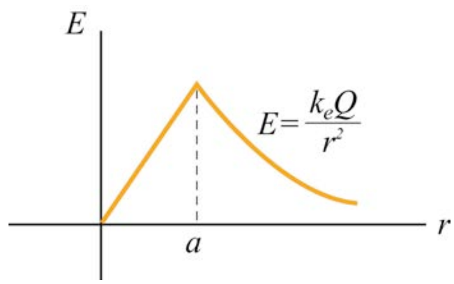


Figure 3.2.16 Electric field due to a uniformly charged sphere as a function of r.

3.3 Conductors

An insulator such as glass or paper is a material in which electrons are attached to some particular atoms and cannot move freely. On the other hand, inside a conductor, electrons are free to move around. The basic properties of a conductor in electrostatic equilibrium are as follows.

(1) The electric field is zero inside a conductor.

If we place a solid spherical conductor in a constant external field $\vec{\mathbf{E}}_0$, the positive and negative charges will move toward the polar regions of the sphere (the regions on the left and right of the sphere in Figure 3.3.1 below), thereby inducing an electric field $\vec{\mathbf{E}}'$. Inside the conductor, $\vec{\mathbf{E}}'$ points in the opposite direction of $\vec{\mathbf{E}}_0$. Since charges are mobile, they will continue to move until $\vec{\mathbf{E}}'$ completely cancels $\vec{\mathbf{E}}_0$ inside the conductor. At electrostatic equilibrium, $\vec{\mathbf{E}}$ must vanish inside a conductor. Outside the conductor, the electric field $\vec{\mathbf{E}}'$ due to the induced charge distribution corresponds to a dipole field, and the total electrostatic field is simply $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}'$. The field lines are depicted in Figure 3.3.1.

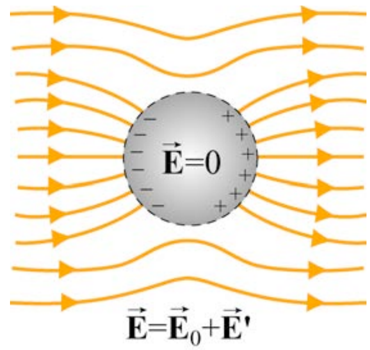


Figure 3.3.1 Placing a conductor in a uniform electric field $\vec{\mathbf{E}}_0$.

(2) Any net charge must reside on the surface.

If there were a net charge inside the conductor, then by Gauss's law (Eq. (3.2.5)), $\vec{\mathbf{E}}$ would no longer be zero there. Therefore, all the net excess charge must flow to the surface of the conductor.

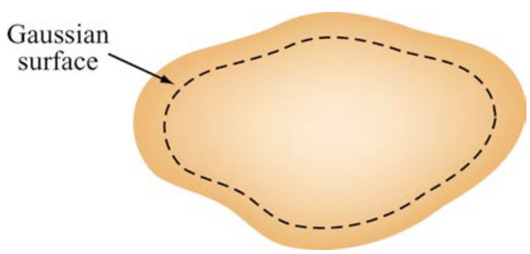


Figure 3.3.2 Gaussian surface inside a conductor. The enclosed charge is zero.

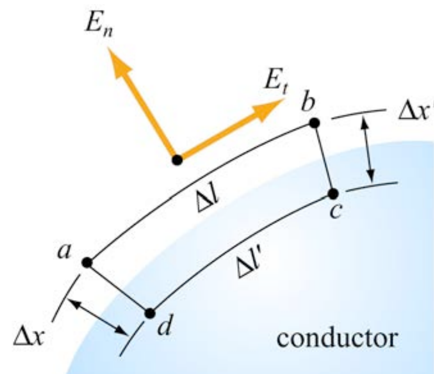


Figure 3.3.3 Normal and tangential components of electric field outside the conductor

(3) The tangential component of $\vec{\mathbf{E}}$ is zero on the surface of a conductor.

We have already seen that for an isolated conductor, the electric field is zero in its interior. Any excess charge placed on the conductor must then distribute itself on the surface, as implied by Gauss's law.

Consider the line integral $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ around a closed path shown in Figure 3.3.3. Because the electrostatic field $\vec{\mathbf{E}}$ is conservative, the line integral around the closed path *abcda* vanishes:

$$\oint_{abcda} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_t(\Delta l) - E_n(\Delta x') + 0(\Delta l') + E_n(\Delta x) = 0,$$

where E_t and E_n are the tangential and the normal components of the electric field, respectively, and we have oriented the segment *ab* so that it is parallel to E_t. In the limit where both Δx and $\Delta x' \to 0$, we have $E_t \Delta l = 0$. Because the length element Δl is finite, we conclude that the tangential component of the electric field on the surface of a conductor vanishes:

$$E_t = 0 \qquad \text{(on the surface of a conductor).} \tag{3.3.1}$$

(We shall see in Chapter 4 that this property that the tangential component of the electric field on the surface is zero implies that the surface of a conductor in electrostatic equilibrium is an *equipotential surface*.)

(4) $\vec{\mathbf{E}}$ is normal to the surface just outside the conductor.

If the tangential component of $\vec{\mathbf{E}}$ is initially non-zero, charges will then move around until it vanishes. Hence, only the normal component survives.

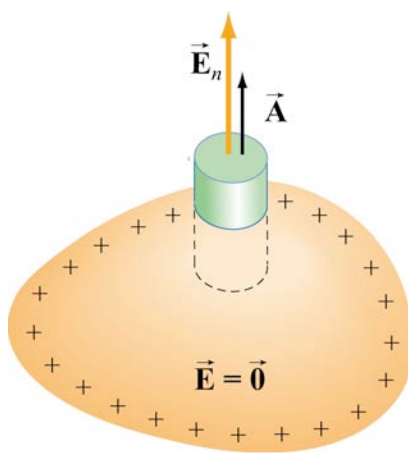


Figure 3.3.3 Gaussian "pillbox" for computing the electric field outside the conductor.

To compute the field strength just outside the conductor, consider the Gaussian pillbox drawn in Figure 3.3.3. Using Gauss's law, we obtain

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_n A + (0)(A) = \frac{\sigma A}{\varepsilon_0}. \tag{3.3.2}$$

Therefore the normal component of the electric field is proportional to the surface charge density

$$E_n = \frac{\sigma}{\varepsilon_0}. \tag{3.3.3}$$

The above result holds for a conductor of arbitrary shape. The pattern of the electric field line directions for the region near a conductor is shown in Figure 3.3.4.

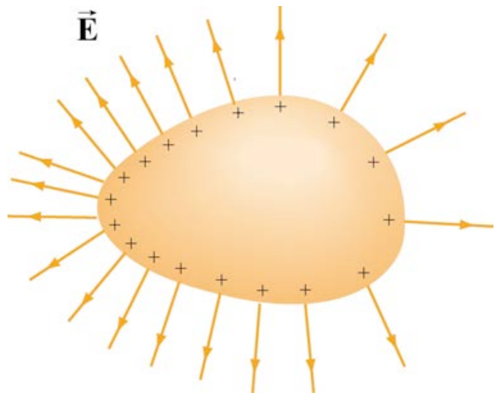


Figure 3.3.4 Just outside a conductor, $\vec{\mathbf{E}}$ is always perpendicular to the surface.

(5) As in the examples of an infinitely large non-conducting plane and a spherical shell, the normal component of the electric field exhibits a discontinuity at the boundary:

$$\Delta E_n = E_n^{(+)} - E_n^{(-)} = \frac{\sigma}{\varepsilon_0} - 0 = \frac{\sigma}{\varepsilon_0}. \tag{3.3.4}$$

Example 3.5: Conductor with Charge Inside a Cavity

Consider a hollow conductor shown in Figure 3.3.5 below. Suppose the net charge carried by the conductor is $+Q$. In addition, there is a charge q inside the cavity. What is the charge on the outer surface of the conductor?

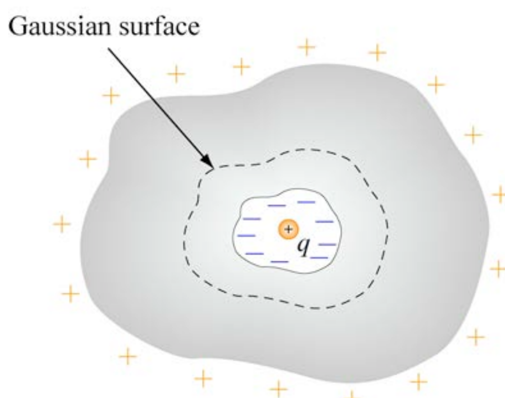


Figure 3.3.5 Conductor with a cavity

Since the electric field inside a conductor must be zero, the net charge enclosed by the Gaussian surface shown in Figure 3.3.5 must be zero. This implies that a charge $-q$ must have been induced on the cavity surface. Since the conductor itself has a charge $+Q$, the amount of charge on the outer surface of the conductor must be $Q+q$.

3.4 Force on a Conductor

We have seen that at the boundary surface of a conductor with a uniform charge density σ, the tangential component of the electric field is zero, and hence, continuous, while the normal component of the electric field exhibits discontinuity, with $\Delta E_n = \sigma / \varepsilon_0$. Consider a small patch of charge on a conducting surface, as shown in Figure 3.4.1.

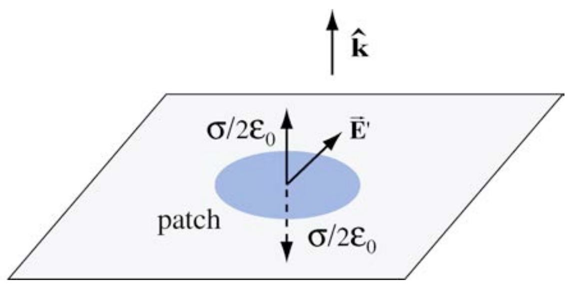


Figure 3.4.1 Force on a conductor

What is the force experienced by this patch? To answer this question, let's write the electric field anywhere outside the surface as

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{\text{patch}} + \vec{\mathbf{E}}', \tag{3.4.1}$$

where $\vec{\mathbf{E}}_{\text{patch}}$ is the electric field due to the charge on the patch, and $\vec{\mathbf{E}}'$ is the electric field due to all other charges. Since by Newton's third law, the patch cannot exert a force on itself, the force on the patch must come solely from $\vec{\mathbf{E}}'$. Assuming the patch to be a flat surface, from Gauss's law, the electric field due to the patch is

$$\vec{\mathbf{E}}_{\text{patch}} = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < 0. \end{cases} \tag{3.4.2}$$

By superposition principle, the electric field above the conducting surface is

$$\vec{\mathbf{E}}_{\text{above}} = \left(\frac{\sigma}{2\varepsilon_0}\right)\hat{\mathbf{k}} + \vec{\mathbf{E}}'. \tag{3.4.3}$$

Similarly, below the conducting surface, the electric field is

$$\vec{\mathbf{E}}_{\text{below}} = -\left(\frac{\sigma}{2\varepsilon_0}\right)\hat{\mathbf{k}} + \vec{\mathbf{E}}'. \tag{3.4.4}$$

The electric field $\vec{\mathbf{E}}'$ is continuous across the boundary. This is due to the fact that if the patch were removed, the field in the remaining "hole" exhibits no discontinuity. Using the two equations above, we find

$$\vec{\mathbf{E}}' = \frac{1}{2}\left(\vec{\mathbf{E}}_{\text{above}} + \vec{\mathbf{E}}_{\text{below}}\right) = \vec{\mathbf{E}}_{\text{avg}}. \tag{3.4.5}$$

In the case of a conductor, with $\vec{\mathbf{E}}_{\text{above}} = (\sigma / \varepsilon_0)\hat{\mathbf{k}}$ and $\vec{\mathbf{E}}_{\text{below}} = 0$, we have

$$\vec{\mathbf{E}}_{\text{avg}} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon_0}\hat{\mathbf{k}} + 0\right) = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}. \tag{3.4.6}$$

Thus, the force acting on the patch is

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}}_{\text{avg}} = (\sigma A)\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} = \frac{\sigma^2 A}{2\varepsilon_0}\hat{\mathbf{k}}, \tag{3.4.7}$$

where A is the area of the patch. This is precisely the force needed to drive the charges on the surface of a conductor to an equilibrium state where the electric field just outside the conductor takes on the value σ / ε_0 and vanishes inside. Note that irrespective of the sign of σ, the force tends to pull the patch into the field.

Using the result obtained above, we may define the electrostatic tension on the patch (which for other field configurations is a pressure, see Section 2.11.5) as

$$P = \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{1}{2}\varepsilon_0 E^2, \tag{3.4.8}$$

where E is the magnitude of the field just above the patch. The tension is being transmitted via the electric field.

3.5 Summary

- The **electric flux** that passes through a surface characterized by the area vector $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$ is

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} = EA\cos\theta$$

where θ is the angle between the electric field $\vec{\mathbf{E}}$ and the unit vector $\hat{\mathbf{n}}$.

- In general, the electric flux through a surface is

$$\Phi_E = \iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

- **Gauss's law** states that the electric flux through any closed Gaussian surface is proportional to the total charge enclosed by the surface:

$$\Phi_E = \oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

  Gauss's law can be used to calculate the electric field for a system that possesses planar, cylindrical or spherical symmetry.

- The normal component of the electric field exhibits discontinuity, with $\Delta E_n = \sigma / \varepsilon_0$, when crossing a boundary with surface charge density σ.

- The basic properties of a **conductor** are (1) The electric field inside a conductor is zero; (2) any net charge must reside on the surface of the conductor; (3) the tangential component of the electric field on the surface is zero; (4) just outside the conductor, the electric field is normal to the surface; and (5) the discontinuity in the normal component of the electric field across the surface of a conductor is proportional to the surface charge density.

- **Electrostatic tension** on a conducting surface is

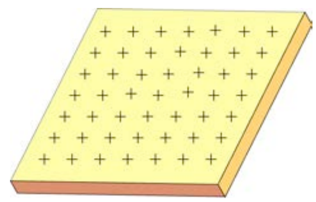
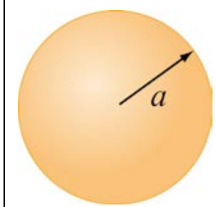
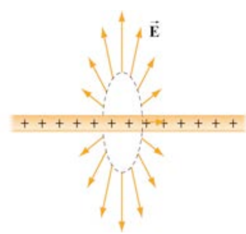
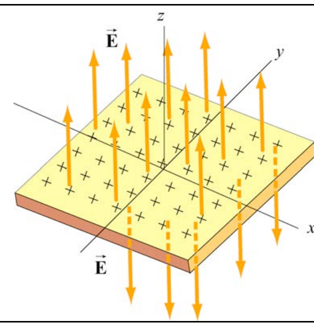
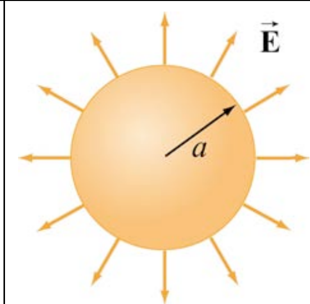
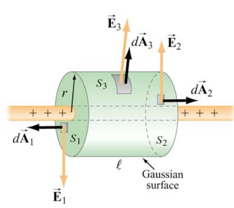
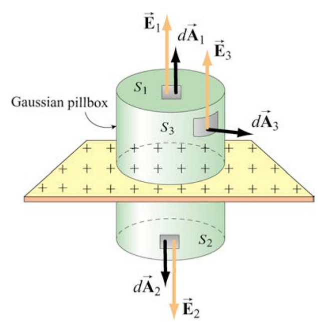
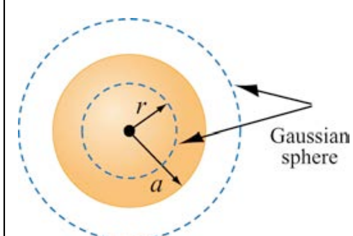
$$P = \frac{F}{A} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{1}{2}\varepsilon_0 E^2 .$$

3.6 Problem-Solving Strategies

In this chapter, we have shown how electric field can be computed using Gauss's law:

$$\Phi_E = \oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

The procedures are outlined in Section 3.2. Below we summarize how the above procedures can be employed to compute the electric field for a line of charge, an infinite plane of charge and a uniformly charged solid sphere.

System	**Infinite line of charge**	**Infinite plane of charge**	**Uniformly charged solid sphere**
			
Identify the symmetry	Cylindrical	Planar	Spherical
Determine the direction of $\vec{\mathbf{E}}$			
Divide the space into different regions	$r > 0$	$z > 0$ and $z < 0$	$r \le a$ and $r \ge a$
Choose Gaussian surface	 Coaxial cylinder	 Gaussian pillbox	 Concentric sphere
Calculate electric flux	$\Phi_E = E(2\pi r l)$	$\Phi_E = EA + EA = 2EA$	$\Phi_E = E(4\pi r^2)$
Calculate enclosed charge q_{in}	$q_{\text{enc}} = \lambda l$	$q_{\text{enc}} = \sigma A$	$q_{\text{enc}} = \begin{cases} Q(r/a)^3 & r \le a \\ Q & r \ge a \end{cases}$
Apply Gauss's law $\Phi_E = q_{\text{in}} / \varepsilon_0$ to find E	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$	$E = \dfrac{\sigma}{2\varepsilon_0}$	$E = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_0 a^3}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r \ge a \end{cases}$

3.7 Solved Problems

3.7.1 Two Parallel Infinite Non-Conducting Planes

Two parallel infinite non-conducting planes lying in the xy-plane are separated by a distance d. Each plane is uniformly charged with equal but opposite surface charge densities, as shown in Figure 3.7.1. Find the electric field everywhere in space.

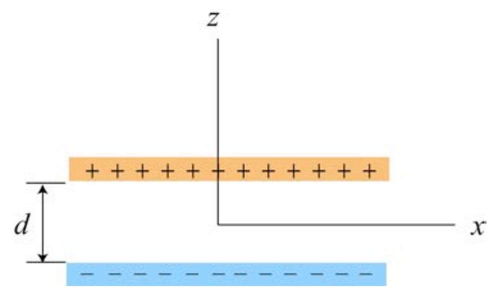


Figure 3.7.1 Positive and negative uniformly charged infinite planes

Solution: The electric field due to the two planes can be found by applying the superposition principle to the result obtained in Example 3.2 for one plane. Since the planes carry equal but opposite surface charge densities, both fields have equal magnitude:

$$E_+ = E_- = \frac{\sigma}{2\varepsilon_0}.$$

The field of the positive plane points away from the positive plane and the field of the negative plane points towards the negative plane (Figure 3.8.2)

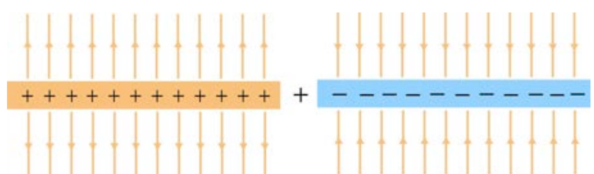


Figure 3.7.2 Electric field of positive and negative planes

Therefore, when we add these fields together, we see that the field outside the parallel planes is zero, and the field between the planes has twice the magnitude of the field of either plane.

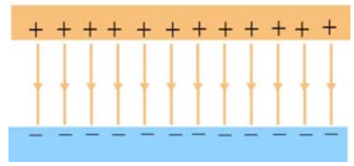


Figure 3.7.3 Electric field of two parallel planes

The electric field of the positive and the negative planes are given by

$$\vec{\mathbf{E}}_+ = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > d/2 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < d/2 \end{cases} , \qquad \vec{\mathbf{E}}_- = \begin{cases} -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > -d/2 \\ +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < -d/2. \end{cases}$$

Adding these two fields together then yields

$$\vec{\mathbf{E}} = \begin{cases} 0\,\hat{\mathbf{k}}, & z > d/2 \\ -\dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{k}}, & d/2 > z > -d/2 \\ 0\,\hat{\mathbf{k}}, & z < -d/2. \end{cases} \qquad (3.7.1)$$

Note that the magnitude of the electric field between the plates is $E = \sigma/\varepsilon_0$, which is twice that of a single plate, and vanishes in the regions $z > d/2$ and $z < -d/2$.

3.7.2 Electric Flux Through a Square Surface

(a) Compute the electric flux through a square surface of edges $2l$ due to a charge $+Q$ located at a perpendicular distance l from the center of the square, as shown in Figure 3.7.4.

(b) Using the result obtained in (a), if the charge $+Q$ is now at the center of a cube of side $2l$ (Figure 3.7.5), what is the total flux emerging from all the six faces of the closed surface?

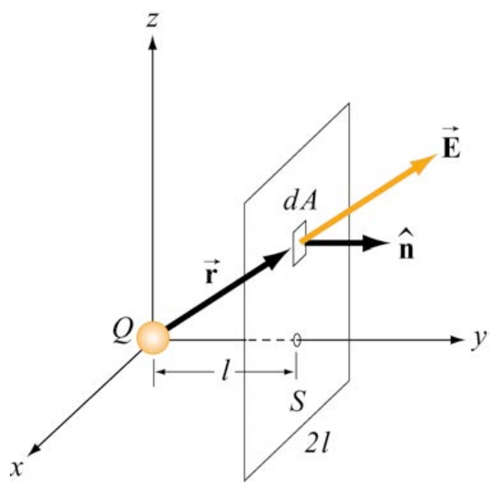


Figure 3.7.4 Electric flux through a square surface

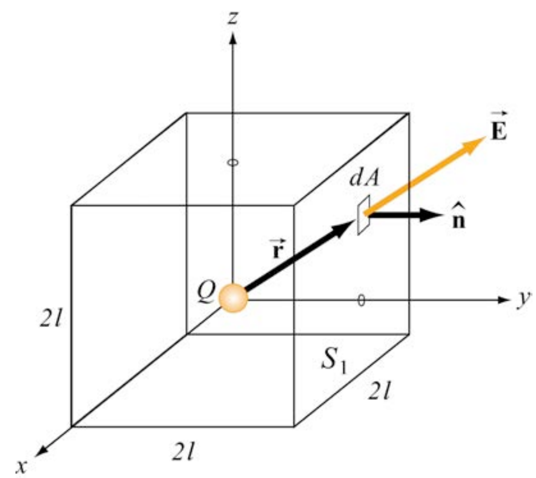


Figure 3.7.5 Electric flux through the surface of a cube

Solutions:

(a) The electric field due to the charge $+Q$ is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}\right),$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ in Cartesian coordinates. On the surface S, $y = l$ and the area element is $d\vec{\mathbf{A}} = dA\hat{\mathbf{j}} = (dx\,dz)\hat{\mathbf{j}}$. Because $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0$ and $\hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = 1$, we have

$$\vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}\right)\cdot(dx\,dz)\hat{\mathbf{j}} = \frac{Ql}{4\pi\varepsilon_0 r^3}dx\,dz\,.$$

Thus, the electric flux through S is

$$\begin{aligned}
\Phi_E &= \iint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l}dx\int_{-l}^{l}\frac{dz}{(x^2+l^2+z^2)^{3/2}} = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l}dx\left.\frac{z}{(x^2+l^2)(x^2+l^2+z^2)^{1/2}}\right|_{-l}^{l} \\
&= \frac{Ql}{2\pi\varepsilon_0}\int_{-l}^{l}\frac{l\,dx}{(x^2+l^2)(x^2+2l^2)^{1/2}} = \frac{Q}{2\pi\varepsilon_0}\tan^{-1}\left.\left(\frac{x}{\sqrt{x^2+2l^2}}\right)\right|_{-l}^{l} \\
&= \frac{Q}{2\pi\varepsilon_0}\left[\tan^{-1}(1/\sqrt{3}) - \tan^{-1}(-1/\sqrt{3})\right] = \frac{Q}{6\varepsilon_0},
\end{aligned}$$

where the following integrals have been used:

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$$
$$\int \frac{dx}{(x^2+a^2)(x^2+b^2)^{1/2}} = \frac{1}{a(b^2-a^2)^{1/2}} \tan^{-1}\left(x\sqrt{\frac{(b^2-a^2)}{a^2(x^2+b^2)}} \right), \; b^2 > a^2.$$

(b) From symmetry arguments, the flux through each face must be the same. Thus, the total flux through the cube is just six times that through one face:

$$\Phi_E = 6\left(\frac{Q}{6\varepsilon_0}\right) = \frac{Q}{\varepsilon_0}.$$

The result shows that the electric flux Φ_E passing through a closed surface is proportional to the charge enclosed. In addition, the result further reinforces the notion that Φ_E is independent of the shape of the closed surface.

3.7.3 Gauss's Law for Gravity

What is the gravitational field inside a spherical shell of radius a and mass m?

Solution: Because the gravitational force is also an inverse square law, there is an equivalent Gauss's law for gravitation:

$$\Phi_g = -4\pi G m_{\text{enc}}. \tag{3.7.2}$$

The only changes are that we calculate gravitational flux, the constant $1/\varepsilon_0 \to -4\pi G$, and $q_{\text{enc}} \to m_{\text{enc}}$. For $r \le a$, the mass enclosed in a Gaussian surface is zero because the mass is all on the shell. Therefore the gravitational flux on the Gaussian surface is zero. This means that the gravitational field inside the shell is zero!

3.8 Conceptual Questions

1. If the electric field in some region of space is zero, does it imply that there is no electric charge in that region?

2. Consider the electric field due to a non-conducting infinite plane having a uniform charge density. Why is the electric field independent of the distance from the plane? Explain in terms of the spacing of the electric field lines.

3. If we place a point charge inside a hollow sealed conducting pipe, describe the electric field outside the pipe.

4. Consider two isolated spherical conductors each having net charge $Q > 0$. The spheres have radii *a* and *b*, where $b > a$. Which sphere has the higher potential?

3.9 Additional Problems

3.9.1 Non-Conducting Solid Sphere with a Cavity

A sphere of radius 2*R* is made of a non-conducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius *R* is then carved out from the sphere, as shown in the figure below. Compute the electric field within the cavity.

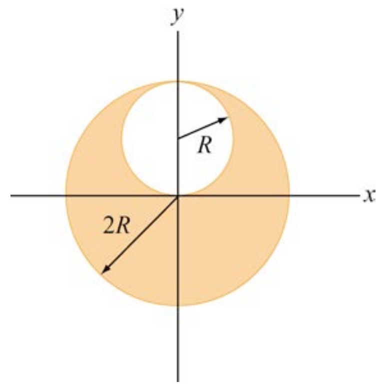


Figure 3.9.1 Non-conducting solid sphere with a cavity

3.9.2 Thin Slab

Let some charge be uniformly distributed throughout the volume of a large planar slab of plastic of thickness d. The charge density is ρ. The mid-plane of the slab is the yz-plane.

(a) What is the electric field at a distance x from the mid-plane where $|x| < d/2$?

(b) What is the electric field at a distance x from the mid-plane where $|x| > d/2$?
[Hint: put part of your Gaussian surface where the electric field is zero.]

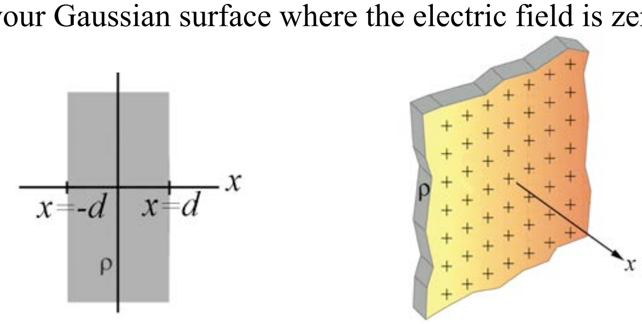
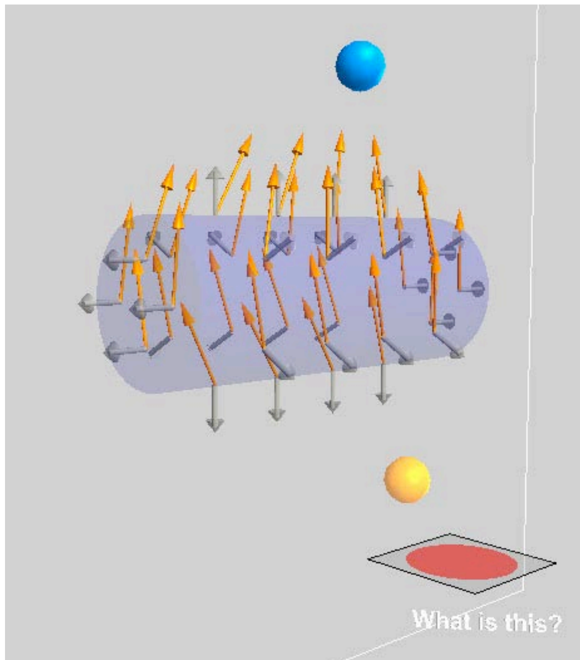


Figure 3.9.2 Thin solid charged infinite slab

3.10 Gauss's Law Simulation

In this section we explore the meaning of Gauss's Law using a 3D simulation that creates imaginary, moveable Gaussian surfaces in the presence of real, moveable point charges. This simulation illustrates Gauss's Law for a spherical or cylindrical imaginary Gaussian surface, in the presence of positive (orange) or negative (blue) point charges.



http://peter-edx.99k.org/GaussLawFlux.html

Figure 3.10.1 Screen Shot of Gauss's Law Simulation

You begin with one positive charge and one negative charge in the scene. You can add additional positive or negative charges, or delete all charges present and start again. Left clicking and dragging on the charge can move the charges. You can choose whether your imaginary closed Gaussian surface is a cylinder or a sphere, and you can move that surface. You will see normals to the Gaussian surface (gray arrows) at many points on the surface. At those same points you will see the local electric field (yellow vectors) at that point on the surface due to all the charges in the scene. If you left click and drag in the view, your perspective will change so that you can see the field vector and normal orientation better. If you want to return to the original view you can "Reset Camera."

Use the simulation to verify the following properties of Gauss Law. For the closed surface, you may choose either a Gaussian cylinder or a Gaussian sphere.

(1) If charges are placed outside a Gaussian surface, the total electric flux through that closed surface is zero.

(2) If a charge is placed inside a Gaussian surface, the total electric flux through that closed surface is positive or negative depending on the sign of the charge enclosed.

(3) If more than one charge is inside the Gaussian surface, the total electric flux through that closed surface depends on the number and sign of the charges enclosed.

Then use the simulation to answer the two following questions. Consider two charged objects. Place one of the charged objects ***inside*** your closed Gaussian surface and the other ***outside***.

(1) Is the electric field on the closed surface due only to the charged objects that are inside that surface?

(2) Is the electric flux on a given part of the Gaussian surface due only to the charged objects that are inside that surface?

(3) Is the *total* electric flux through the entire Gaussian surface due only to the charged objects that are inside that surface?