Chapter 2

Coulomb's Law

Coulomb's Law

2.1 Electric Charge

There are two types of observed electric charge, which we designate as positive and negative. The convention was derived from Benjamin Franklin's experiments. He rubbed a glass rod with silk and called the charges on the glass rod positive. He rubbed sealing wax with fur and called the charge on the sealing wax negative. Like charges repel and opposite charges attract each other. The unit of charge is called the Coulomb (C).

The smallest unit of "free" charge known in nature is the charge of an electron or proton, which has a magnitude of

$$e = 1.602 \times 10^{-19}\,\mathrm{C}. \tag{2.1.1}$$

Charge of any ordinary matter is quantized in integral multiples of e. An electron carries one unit of negative charge, $-e$, while a proton carries one unit of positive charge, $+e$. In a closed system, the total amount of charge is conserved since charge can neither be created nor destroyed. A charge can, however, be transferred from one body to another.

2.2 Coulomb's Law

In this section we will simply state Coulomb's Law. This is the path followed by almost all introductory textbooks. But you will get a much better understanding of this law at an intuitive level in reading Section 2.11 below, where we explain how Faraday thought of this law, in terms of his lines of force (see also Section 1.1).

Consider a system of two point-like objects with charges, q_1 and q_2, separated by a distance r in vacuum. The electric force exerted by q_1 on q_2 is given by Coulomb's law,

$$\vec{\mathbf{F}}_{12} = k_e \frac{q_1 q_2}{r^2}\hat{\mathbf{r}}, \tag{2.2.1}$$

where k_e is the Coulomb constant, and $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$ is a unit vector directed from q_1 to q_2 as illustrated in Figure 2.2.1(a). Similarly, the force on q_1 due to q_2 is given by $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$, as illustrated in Figure 2.2.1(b). This is consistent with Newton's third law.

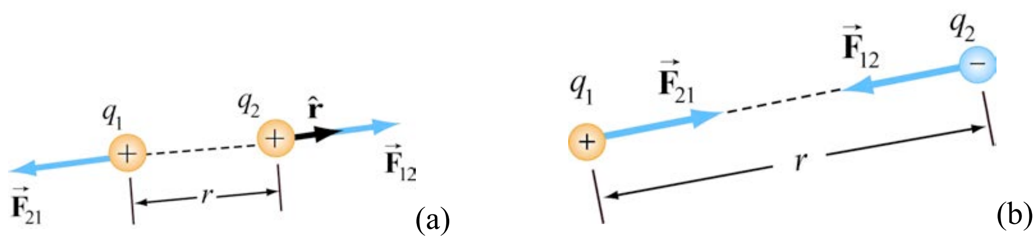


Figure 2.2.1 Coulomb interaction between two charges

Note that electric force is a vector that has both magnitude and direction. In SI units, the Coulomb constant k_e is given by

$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.9875\times 10^9\,\mathrm{N\cdot m^2/C^2}, \tag{2.2.2}$$

where ε_0 is the electric constant called the *permittivity of free space*. The value of ε_0 is exactly equal to

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}, \tag{2.2.3}$$

where the constant $\mu_0 = 4\pi\times 10^{-9}\ \mathrm{N\cdot s^2\cdot C^{-2}}$ is called the *permeability of free space*, and $c = 299792458\,\mathrm{m\cdot s^{-1}}$ is the speed of light. Therefore

$$\begin{aligned}\varepsilon_0 = \frac{1}{\mu_0 c^2} &= \frac{1}{(4\pi\times 10^{-9}\ \mathrm{N\cdot s^2\cdot C^{-2}})(299792458\,\mathrm{m\cdot s^{-1}})^2} \\ &= 8.854187817...\times 10^{-12}\ \mathrm{C^2/N\cdot m^2}.\end{aligned} \tag{2.2.4}$$

Before the speed of light was exactly defined to be $c = 299792458\,\mathrm{m\cdot s^{-1}}$, the value of ε_0 depended on the experimentally measured value of the speed of light. Now both ε_0 and μ_0 are defined exactly.

2.3 Principle of Superposition

Coulomb's law applies to any pair of point charges. When more than two charges are present, the net force on any one charge is simply the vector sum of the forces exerted on it by the other charges. For example, if three charges are present, the resultant force experienced by q_3 due to q_1 and q_2 will be

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23}. \tag{2.3.1}$$

The superposition principle is illustrated in the example below.

Example 2.1: Three Charges

Three charges are arranged as shown in Figure 2.3.1. Find the force on the charge q_3 assuming that $q_1 = 6.0\times10^{-6}\ \text{C}$, $q_2 = -q_1 = -6.0\times10^{-6}\ \text{C}$, $q_3 = 3.0\times10^{-6}\ \text{C}$ and $a = 2.0\times10^{-2}\ \text{m}$.

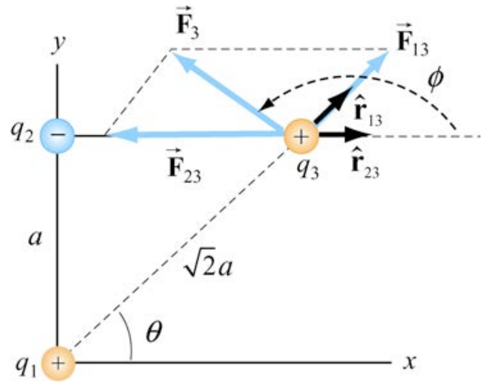


Figure 2.3.1 A system of three charges

Solution: Using the superposition principle, the force on q_3 is

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}^{\ 2}}\hat{\mathbf{r}}_{13} + \frac{q_2 q_3}{r_{23}^{\ 2}}\hat{\mathbf{r}}_{23}\right).$$

In this case the second term will have a negative coefficient, since q_2 is negative. The unit vectors $\hat{\mathbf{r}}_{13}$ and $\hat{\mathbf{r}}_{23}$ do not point in the same directions. In order to compute this sum, we can express each unit vector in terms of its Cartesian components and add the forces according to the principle of vector addition.

From the figure, we see that the unit vector $\hat{\mathbf{r}}_{13}$ which points from q_1 to q_3 can be written as

$$\hat{\mathbf{r}}_{13} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} = \frac{\sqrt{2}}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}}).$$

Similarly, the unit vector $\hat{\mathbf{r}}_{23} = \hat{\mathbf{i}}$ points from q_2 to q_3 . Therefore upon adding the components, the total force is

$$\vec{\mathbf{F}}_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}^2}\hat{\mathbf{r}}_{13} + \frac{q_2 q_3}{r_{23}^2}\hat{\mathbf{r}}_{23}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{(\sqrt{2}a)^2}\frac{\sqrt{2}}{2}(\hat{\mathbf{i}}+\hat{\mathbf{j}}) + \frac{(-q_1)q_3}{a^2}\hat{\mathbf{i}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{a^2}\left[\left(\frac{\sqrt{2}}{4}-1\right)\hat{\mathbf{i}} + \frac{\sqrt{2}}{4}\hat{\mathbf{j}}\right].$$

The magnitude of the total force is given by

$$F_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{a^2}\left[\left(\frac{\sqrt{2}}{4}-1\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2\right]^{1/2}$$

$$= (9.0\times 10^9\,\mathrm{N\cdot m^2/C^2})\frac{(6.0\times 10^{-6}\,\mathrm{C})(3.0\times 10^{-6}\,\mathrm{C})}{(2.0\times 10^{-2}\,\mathrm{m})^2}(0.74) = 3.0\times 10^2\,\mathrm{N}.$$

The angle that the force makes with the positive x-axis is

$$\phi = \tan^{-1}\left(\frac{F_{3,y}}{F_{3,x}}\right) = \tan^{-1}\left[\frac{\sqrt{2}/4}{-1+\sqrt{2}/4}\right] = 151.3^\circ.$$

Note there are two solutions to this equation. The second solution $\phi = 28.7^\circ$ is incorrect because it would indicate that the force has positive $\hat{\mathbf{i}}$- and negative $\hat{\mathbf{j}}$-components.

For a system of N charges, the net force experienced by the jth particle would be

$$\vec{\mathbf{F}}_j = \sum_{\substack{i=1\\ i\neq j}}^{N}\vec{\mathbf{F}}_{ij}, \tag{2.3.2}$$

where $\vec{\mathbf{F}}_{ij}$ denotes the force between particles i and j. The superposition principle implies that the net force between any two charges is independent of the presence of other charges. This is true if the charges are in fixed positions.

2.4 Electric Field

The electrostatic force, like the gravitational force, is a force that acts at a distance, even when the objects are not in contact with one another. To justify such a notion we rationalize action at a distance by saying that one charge creates a field that in turn acts on the other charge.

An electric charge q produces an electric field everywhere. To quantify the strength of the field created by that charge, we can measure the force a positive "test charge" q_0 experiences at some point. The electric field $\vec{\mathbf{E}}$ is defined as:

$$\vec{\mathbf{E}} = \lim_{q_0 \to 0} \frac{\vec{\mathbf{F}}_e}{q_0} . \tag{2.4.1}$$

We take q_0 to be infinitesimally small so that the field q_0 generates does not disturb the "source charges." The analogy between the electric field and the gravitational field $\vec{\mathbf{g}} = \lim_{m_0 \to 0} \vec{\mathbf{F}}_m / m_0$ is depicted in Figure 2.4.1.

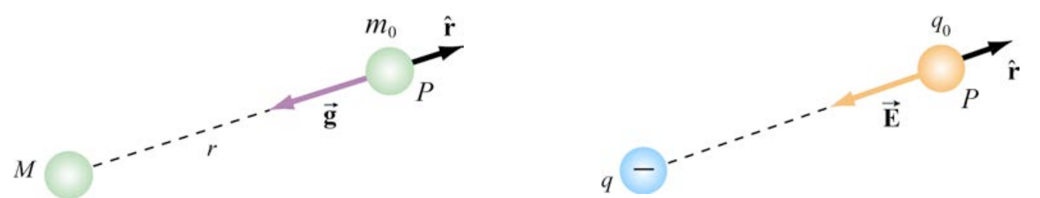


Figure 2.4.1 Analogy between the gravitational field $\vec{\mathbf{g}}$ and the electric field $\vec{\mathbf{E}}$.

From the field theory point of view, we say that the charge q creates an electric field $\vec{\mathbf{E}}$ that exerts a force $\vec{\mathbf{F}}_e = q_0\vec{\mathbf{E}}$ on a test charge q_0.

Using the definition of electric field given in Eq. (2.4.1) and the Coulomb's law, the electric field at a distance r from a point charge q is given by

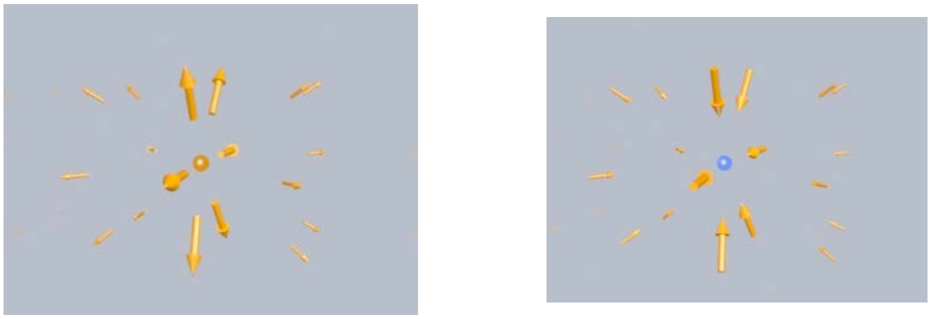
$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} . \tag{2.4.2}$$

Using the superposition principle, the total electric field due to a group of charges is equal to the vector sum of the electric fields of individual charges:

$$\vec{\mathbf{E}} = \sum_i \vec{\mathbf{E}}_i = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i^2} \hat{\mathbf{r}} . \tag{2.4.3}$$

2.4.1 Electric Field of Point Charges

Figure 2.4.2 shows one frame of movies of the electric field of a moving positive and a moving negative point charge, assuming the speed of the charge is small compared to the speed of light.



(a) link1 (b) link2

Figure 2.4.2 The electric fields of (a) a moving positive charge, (b) a moving negative charge, when the speed of the charge is small compared to the speed of light.

2.5 Electric Field Lines

Electric field lines provide a convenient graphical representation of the electric field in space. The field lines for a stationary positive and a stationary negative charge are shown in Figure 2.5.1.

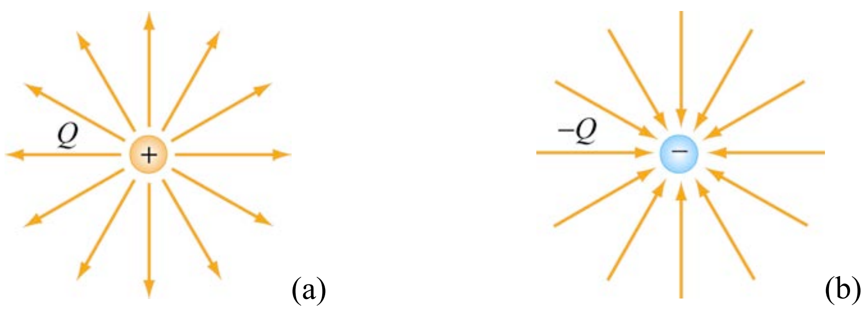


Figure 2.5.1 Field lines for (a) positive and (b) negative charges.

Notice that the direction of field lines is radially outward for a positive charge and radially inward for a negative charge. For a pair of charges of equal magnitude but opposite sign (an electric dipole), the field lines are shown in Figure 2.5.2.

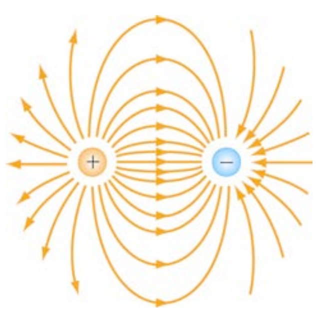


Figure 2.5.2 Field lines for a finite electric dipole.

The pattern of electric field lines can be obtained by considering the following.

(1) Symmetry: for every point above the line joining the two charges there is an equivalent point below it. Therefore, the pattern must be symmetrical about the line joining the two charges.

(2) Near field: very close to a charge, the field due to that charge predominates. Therefore, the lines are radial and spherically symmetric.

(3) Far field: far from the system of charges, the pattern should look like that of a single point charge of value $Q = \sum_i Q_i$. Thus, the lines should be radially inward or outward, unless $Q = 0$.

(4) Null point: This is a point at which $\vec{\mathbf{E}} = \vec{\mathbf{0}}$, and no field lines should pass through it.

The properties of electric field lines may be summarized as follows:

- The direction of the electric field vector $\vec{\mathbf{E}}$ at a point is tangent to the field lines.

- The field lines must begin on positive charges (or at infinity) and then terminate on negative charges (or at infinity).

- The number of lines that originate from a positive charge or terminating on a negative charge must be proportional to the magnitude of the charge.

- No two field lines can cross each other; otherwise the field would be pointing in two different directions at the same point.

2.6 Force on a Charged Particle in an Electric Field

Consider a charge $+q$ moving between two parallel plates of opposite charges, as shown in Figure 2.6.1.

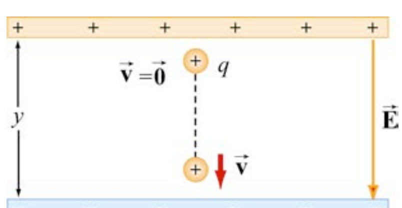


Figure 2.6.1 Charge moving in a constant electric field

Let the electric field between the plates be $\vec{\mathbf{E}} = -E_y\,\hat{\mathbf{j}}$, with $E_y > 0$. (In Chapter 4, we shall show that the electric field in the region between two infinitely large plates of opposite charges is uniform.) The charge will experience a downward Coulomb force

$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}}. \tag{2.6.1}$$

Note the distinction between the charge q that is experiencing a force and the charges on the plates that are the *sources* of the electric field. Even though the charge q is also a source of an electric field, by Newton's third law, the charge cannot exert a force on itself. In Figure 2.6.1 we draw only the electric field lines $\vec{\mathbf{E}}$ due to the "source" charges. This is the standard way the electric field is drawn in this situation in most introductory textbooks (that is, we draw only the field lines of the source charges). That allows us to write down the correct force, since the electric field of the charge experiencing the force does not exert a force on the charge producing it, at least in a static situation, so one might argue that there is no reason to show the total field. However, if we draw the field this way, limiting ourselves to only the source field lines and not the total field lines, we cannot understand the manner in which electric fields transmit forces, following Faraday, as discussed in Section 1.1. To use Faraday's powerful insight, we must draw the total electric field, and we do this and discuss the meaning in Section 2.11.2 below.

In any case, according to Newton's second law, this net force will cause the charge to accelerate with an acceleration given by

$$\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}_e}{m} = \frac{q\vec{\mathbf{E}}}{m} = -\frac{qE_y}{m}\hat{\mathbf{j}}. \tag{2.6.2}$$

Suppose the particle is at rest ($v_0 = 0$) when it is first released from the positive plate. The final speed v of the particle as it strikes the negative plate is

$$v_y = \sqrt{2\,|\,a_y\,|\,y} = \sqrt{\frac{2yqE_y}{m}}, \tag{2.6.3}$$

where y is the distance between the two plates. The kinetic energy of the particle when it strikes the plate is

$$K = \frac{1}{2}mv_y^2 = qE_y y. \tag{2.6.4}$$

You might ask where the kinetic energy of the accelerating charge is coming from, and also its momentum. As we discuss in more detail below, in Section 2.11.2, the kinetic energy going into the charge is coming out of the electric energy stored in the total field at the beginning of this configuration, so that total energy, kinetic plus field energy, is conserved. The momentum of the charge, however, is coming from the momentum of the source charges. That is, the downward momentum gained by the accelerating charge is offset by an upward momentum gained by the source charges, so that total momentum is conserved, but in a different manner (field momentum is not involved).

2.7 Electric Dipole

An electric dipole consists of two equal but opposite charges, $+q$ and $-q$, separated by a distance $2a$, as shown in Figure 2.7.1.

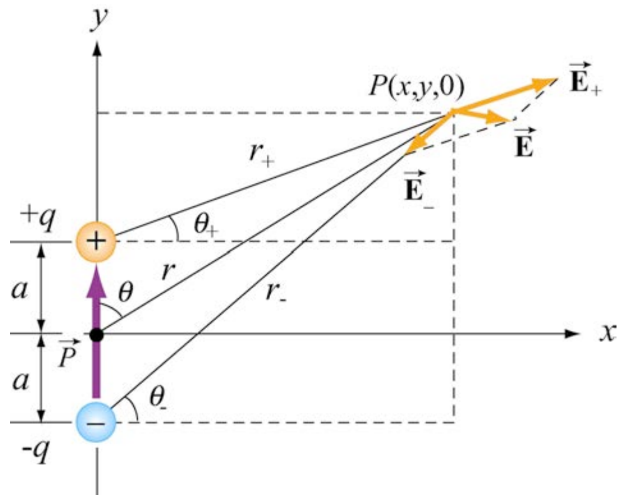


Figure 2.7.1 Electric dipole

By definition, the dipole moment vector $\vec{\mathbf{p}}$ points from $-q$ to $+q$ (in the $+y$- direction) and has magnitude $p = 2qa$, where $q > 0$,

$$\vec{\mathbf{p}} = 2qa\,\hat{\mathbf{j}}. \tag{2.7.1}$$

For an overall charge-neutral system having N charges, the electric dipole vector $\vec{\mathbf{p}}$ is defined as

$$\vec{\mathbf{p}} \equiv \sum_{i=1}^{i=N} q_i \vec{\mathbf{r}}_i \tag{2.7.2}$$

where $\vec{\mathbf{r}}_i$ is the position vector of the charge q_i. Examples of dipoles include HCL, CO, H_2O and other *polar* molecules. In principle, any molecule in which the centers of the positive and negative charges do not coincide may be approximated as a dipole. In Chapter 5 we shall also show that by applying an external field, an electric dipole moment may also be induced in an unpolarized molecule.

2.7.1 The Electric Field of a Dipole

What is the electric field due to the electric dipole? Referring to Figure 2.7.1, we see that the x-component of the electric field strength at the point P is

$$E_x = \frac{q}{4\pi\varepsilon_0}\left(\frac{\cos\theta_+}{r_+^2} - \frac{\cos\theta_-}{r_-^2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{x}{\left[x^2+(y-a)^2\right]^{3/2}} - \frac{x}{\left[x^2+(y+a)^2\right]^{3/2}}\right), \quad (2.7.3)$$

where

$$r_\pm^2 = r^2 + a^2 \mp 2ra\cos\theta = x^2 + (y \mp a)^2 . \quad (2.7.4)$$

Similarly, the y-component is

$$E_y = \frac{q}{4\pi\varepsilon_0}\left(\frac{\sin\theta_+}{r_+^2} - \frac{\sin\theta_-}{r_-^2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{y-a}{\left[x^2+(y-a)^2\right]^{3/2}} - \frac{y+a}{\left[x^2+(y+a)^2\right]^{3/2}}\right). \quad (2.7.5)$$

In the "point-dipole" limit where $r \gg a$, one may verify that (see Solved Problem 2.13.4) the above expressions reduce to

$$E_x = \frac{3p}{4\pi\varepsilon_0 r^3}\sin\theta\cos\theta \quad (2.7.6)$$

and

$$E_y = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1) , \quad (2.7.7)$$

where $\sin\theta = x/r$ and $\cos\theta = y/r$. With $3pr\cos\theta = 3\vec{\mathbf{p}}\cdot\vec{\mathbf{r}}$ and some algebra, the electric field may be written as

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\left(-\frac{\vec{\mathbf{p}}}{r^3} + \frac{3(\vec{\mathbf{p}}\cdot\vec{\mathbf{r}})\vec{\mathbf{r}}}{r^5}\right). \quad (2.7.8)$$

Note that Eq. (2.7.8) is valid also in three dimensions where $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. The equation indicates that the electric field $\vec{\mathbf{E}}$ due to a dipole decreases with r as $1/r^3$, unlike the $1/r^2$ behavior for a point charge. This is to be expected since the net charge of a dipole is zero and therefore must fall off more rapidly than $1/r^2$ at large distance. The electric field lines due to a finite electric dipole and a point dipole are shown in Figure 2.7.2.

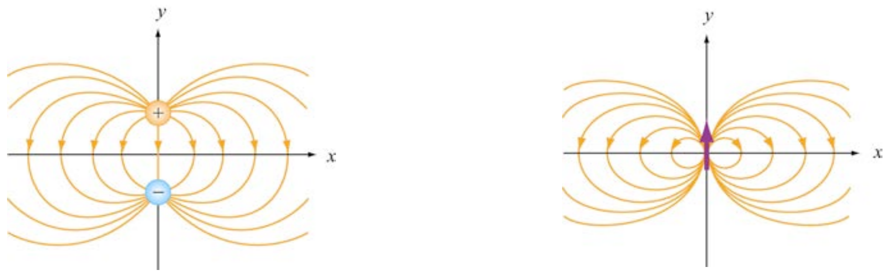


Figure 2.7.2 Electric field lines for (a) a finite dipole and (b) a point dipole.

2.8 Dipole in Electric Field

What happens when we place an electric dipole in a uniform field $\vec{\mathbf{E}} = E\,\hat{\mathbf{i}}$, with the dipole moment vector $\vec{\mathbf{p}}$ making an angle with the x-axis?

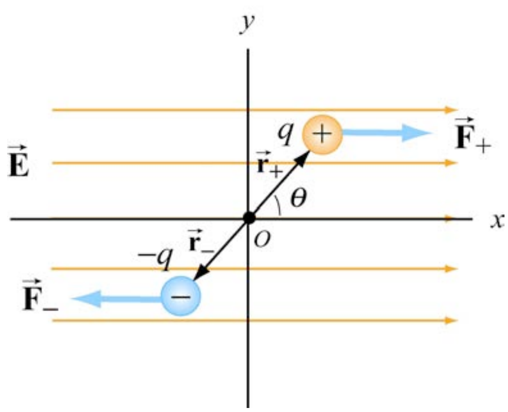


Figure 2.8.1 Electric dipole placed in a uniform field.

From Figure 2.8.1, we see that the unit vector that points in the direction of $\vec{\mathbf{p}}$ is $\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$. Thus, we have

$$\vec{\mathbf{p}} = 2qa(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})\,. \tag{2.8.1}$$

As seen from Figure 2.8.1, because each charge experiences an equal but opposite force due to the field, the net force on the dipole is $\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_+ + \vec{\mathbf{F}}_- = 0$. Even though the net force vanishes, the field exerts a torque a toque on the dipole. The torque about the mid-point of the dipole is

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \vec{\mathbf{r}}_+ \times \vec{\mathbf{F}}_+ + \vec{\mathbf{r}}_- \times \vec{\mathbf{F}}_- = (a\cos\theta\,\hat{\mathbf{i}} + a\sin\theta\,\hat{\mathbf{j}}) \times (F_+\,\hat{\mathbf{i}}) + (-a\cos\theta\,\hat{\mathbf{i}} - a\sin\theta\,\hat{\mathbf{j}}) \times (-F_-\,\hat{\mathbf{i}}) \\ &= a\sin\theta\,F_+(-\hat{\mathbf{k}}) + a\sin\theta\,F_-(-\hat{\mathbf{k}}) \\ &= 2aF\sin\theta(-\hat{\mathbf{k}}),\end{aligned} \tag{2.8.2}$$

where we have used $F_+ = F_- = F$. The direction of the torque is $-\hat{\mathbf{k}}$, or into the page. The effect of the torque $\vec{\boldsymbol{\tau}}_O$ is to rotate the dipole *clockwise* so that the dipole moment $\vec{\mathbf{p}}$ becomes aligned with the electric field $\vec{\mathbf{E}}$. With $F = qE$, the magnitude of the torque can be rewritten as

$$\tau = 2a(qE)\sin\theta = (2aq)E\sin\theta = pE\sin\theta\ ,$$

and the general expression for torque becomes

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}\,. \tag{2.8.3}$$

Thus, we see that the cross product of the dipole moment with the electric field is equal to the torque.

2.8.1 Potential Energy of an Electric Dipole

The work done by the electric field to rotate the dipole by an angle $d\theta$ is

$$dW = -\tau d\theta = -pE\sin\theta\, d\theta\,. \tag{2.8.4}$$

The negative sign indicates that the torque *opposes* any increase in θ. Therefore, the total amount of work done by the electric field to rotate the dipole from an angle θ_0 to θ is

$$W = \int_{\theta_0}^{\theta} (-pE\sin\theta)\, d\theta = pE(\cos\theta - \cos\theta_0)\,. \tag{2.8.5}$$

The result shows that a *positive* work is done by the field when $\cos\theta > \cos\theta_0$. The change in potential energy ΔU of the dipole is the negative of the work done by the field:

$$\Delta U = U - U_0 = -W = -pE(\cos\theta - \cos\theta_0)\,, \tag{2.8.6}$$

We shall choose our zero point for the potential energy when the angle between the dipole moment and the electric field is $\pi/2$, $U(\theta = \pi/2) = 0$. Then, when the dipole moment is at an angle θ with respect to the direction of the external electric field, we define the potential energy function by

$$U(\theta) = -pE\cos\theta = -\vec{\mathbf{p}}\cdot\vec{\mathbf{E}}, \qquad \text{where } U(\pi/2) = 0\,. \tag{2.8.7}$$

A system is at a stable equilibrium when its potential energy is a minimum. This takes place when the dipole $\vec{\mathbf{p}}$ is aligned parallel to $\vec{\mathbf{E}}$, making U a minimum with $U_{\min} = -pE$. On the other hand, when $\vec{\mathbf{p}}$ and $\vec{\mathbf{E}}$ are anti-parallel, $U_{\max} = +pE$ is a maximum and the system is unstable.

If the dipole is placed in a non-uniform field, there would be a net force on the dipole in addition to the torque, and the resulting motion would be a combination of linear acceleration and rotation. In Figure 2.8.2, suppose the electric field $\vec{\mathbf{E}}_+$ at $+q$ differs from the electric field $\vec{\mathbf{E}}_-$ at $-q$.

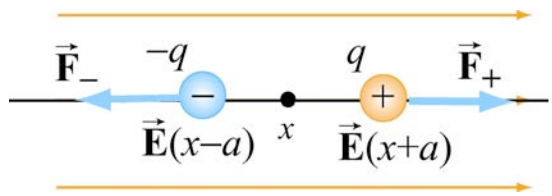


Figure 2.8.2 Force on a dipole

Assuming the dipole to be very small, we expand the fields about x:

$$E_+(x+a) \approx E(x) + a\left(\frac{dE}{dx}\right), \quad E_-(x-a) \approx E(x) - a\left(\frac{dE}{dx}\right). \tag{2.8.8}$$

The force on the dipole then becomes

$$\vec{\mathbf{F}}_e = q(\vec{\mathbf{E}}_+ - \vec{\mathbf{E}}_-) = 2qa\left(\frac{dE}{dx}\right)\hat{\mathbf{i}} = p\left(\frac{dE}{dx}\right)\hat{\mathbf{i}}. \tag{2.8.9}$$

The attraction between small pieces of paper and a comb, which has been charged by rubbing through hair, is an example of non-uniform electric field exerting a force on an electric dipole. The paper has *induced dipole moments* (to be discussed in depth in Chapter 5) while the field on the comb is non-uniform due to its irregular shape (Figure 2.8.3).

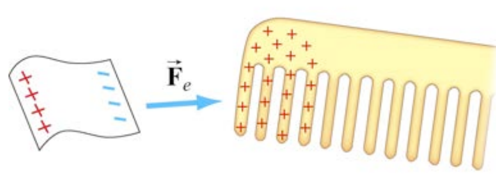


Figure 2.8.3 Electrostatic attraction between a piece of paper and a comb

2.9 Charge Density

The electric field due to a small number of charged particles can readily be computed using the superposition principle. But what happens if we have a very large number of charges distributed in some region in space? Let's consider the system shown in Figure 2.9.1:

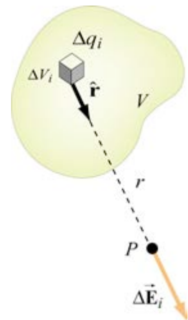


Figure 2.9.1 Electric field due to a small charge element Δq_i.

2.9.1 Volume Charge Density

Suppose we wish to find the electric field at some point P. Let's consider a small volume element ΔV_i that contains an amount of charge Δq_i. The distances between charges within the volume element ΔV_i are much smaller than compared to r, the distance between ΔV_i and P. In the limit where ΔV_i becomes infinitesimally small, we may define a volume charge density $\rho(\vec{\mathbf{r}})$ as

$$\rho(\vec{\mathbf{r}}) = \lim_{\Delta V_i \to 0} \frac{\Delta q_i}{\Delta V_i} = \frac{dq}{dV} . \tag{2.9.1}$$

The dimension of $\rho(\vec{\mathbf{r}})$ is charge/unit volume (C/m^3) in SI units. The total amount of charge within the entire volume V is

$$Q = \sum_i \Delta q_i = \int_V \rho(\vec{\mathbf{r}})\, dV . \tag{2.9.2}$$

The concept of charge density here is analogous to mass density $\rho_m(\vec{\mathbf{r}})$. When a large number of atoms are tightly packed within a volume, we can also take the continuum limit and the mass of an object is given by

$$M = \int_V \rho_m(\vec{\mathbf{r}})\, dV . \tag{2.9.3}$$

2.9.2 Surface Charge Density

In a similar manner, the charge can be distributed over a surface S of area A with a *surface charge density* σ (lowercase Greek letter *sigma*):

$$\sigma(\vec{\mathbf{r}}) = \frac{dq}{dA} . \tag{2.9.4}$$

The dimension of σ is charge/unit area (C/m^2) in SI units. The total charge on the entire surface is:

$$Q = \iint_S \sigma(\vec{\mathbf{r}})\, dA\,. \tag{2.9.5}$$

2.9.3 Line Charge Density

If the charge is distributed over a line of length ℓ, then the *linear charge density* λ (lowercase Greek letter *lambda*) is

$$\lambda(\vec{\mathbf{r}}) = \frac{dq}{d\ell} \tag{2.9.6}$$

where the dimension of λ is charge/unit length (C/m). The total charge is now an integral over the entire length:

$$Q = \int_{\text{line}} \lambda(\vec{\mathbf{r}})\, d\ell\,. \tag{2.9.7}$$

If charges are uniformly distributed throughout the region, the densities $(\rho, \sigma \text{ or } \lambda)$ then become uniform.

2.10 Electric Fields due to Continuous Charge Distributions

The electric field at a point P due to each charge element dq is given by Coulomb's law,

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}}\,, \tag{2.10.1}$$

where r is the distance from dq to P and $\hat{\mathbf{r}}$ is the corresponding unit vector. (See Figure 2.9.1). Using the superposition principle, the total electric field $\vec{\mathbf{E}}$ is the vector sum (integral) of all these infinitesimal contributions:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{dq}{r^2}\hat{\mathbf{r}}\,. \tag{2.10.2}$$

This is an example of a *vector* integral that consists of three separate integrations, one for each component of the electric field.

Example 2.2: Electric Field on the Axis of a Rod

A non-conducting rod of length ℓ with a uniform positive charge density λ and a total charge Q is lying along the x-axis, as illustrated in Figure 2.10.1.

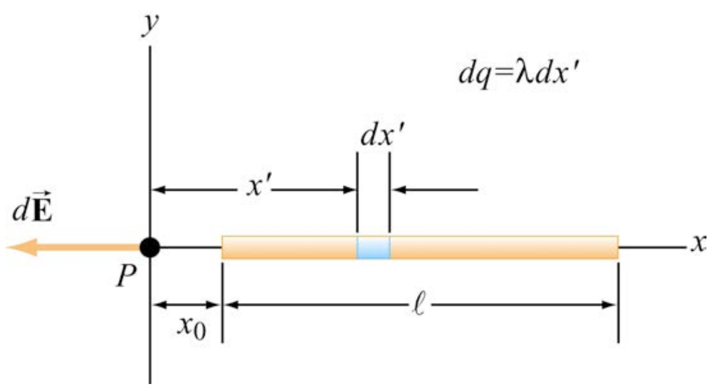


Figure 2.10.1 Electric field of a wire along the axis of the wire

Calculate the electric field at a point P located along the axis of the rod and a distance x_0 from one end.

Solution: The linear charge density is uniform and is given by $\lambda = Q/\ell$. The amount of charge contained in a small segment of length dx' is $dq = \lambda\, dx'$.

The source carries a positive charge Q, the field at P points in the negative x- direction, and $\hat{\mathbf{r}} = -\hat{\mathbf{i}}$ is the unit vector that points from the source to P. Therefore the contribution to the electric field due to dq is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx'}{x'^2}(-\hat{\mathbf{i}}) = -\frac{1}{4\pi\varepsilon_0}\frac{Q dx'}{\ell\, x'^2}\hat{\mathbf{i}}\,.$$

Integrating over the entire length leads to

$$\vec{\mathbf{E}} = \int d\vec{\mathbf{E}} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{\ell}\int_{x_0}^{x_0+\ell}\frac{dx'}{x'^2}\hat{\mathbf{i}} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{\ell}\left(\frac{1}{x_0} - \frac{1}{x_0+\ell}\right)\hat{\mathbf{i}} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0(\ell + x_0)}\hat{\mathbf{i}}\,. \quad (2.10.3)$$

Notice that when P is very far away from the rod, $x_0 \gg \ell$, and the above expression becomes

$$\vec{\mathbf{E}} \approx -\frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0^2}\hat{\mathbf{i}}\,. \quad (2.10.4)$$

The result is to be expected since at sufficiently far distance away, the distinction between a continuous charge distribution and a point charge diminishes.

Example 2.3 Electric Field on the Perpendicular Bisector

A non-conducting rod of length ℓ with a uniform charge density λ and a total charge Q is lying along the x-axis, as illustrated in Figure 2.10.2. Compute the electric field at a point P, located at a distance y from the center of the rod along its perpendicular bisector.

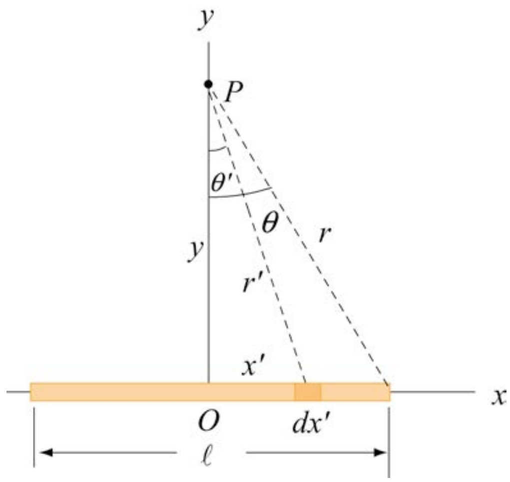


Figure 2.10.2

Solution: We follow a similar procedure as that outlined in Example 2.2. The contribution to the electric field from a small length element dx' carrying charge $dq = \lambda\, dx'$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r'^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2 + y^2}\,. \tag{2.10.5}$$

Using symmetry argument illustrated in Figure 2.10.3, one may show that the x-component of the electric field vanishes.

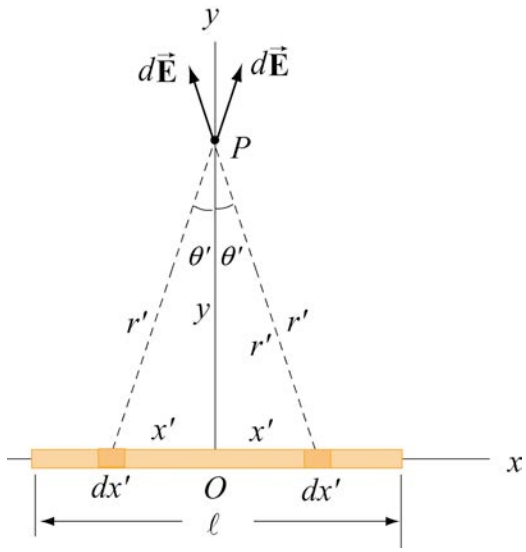


Figure 2.10.3 Symmetry argument showing that $E_x = 0$.

The y-component of dE is

$$dE_y = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx'}{x'^2+y^2}\frac{y}{\sqrt{x'^2+y^2}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda y\,dx'}{(x'^2+y^2)^{3/2}}. \qquad (2.10.6)$$

By integrating over the entire length, the y-component of the electric field due to the rod is

$$E_y = \int dE_y = \frac{1}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{\lambda y dx'}{(x'^2+y^2)^{3/2}} = \frac{\lambda y}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{dx'}{(x'^2+y^2)^{3/2}}. \qquad (2.10.7)$$

By making the change of variable: $x' = y\tan\theta'$, which gives $dx' = y\sec^2\theta'\,d\theta'$, the above integral becomes

$$\begin{aligned}\int_{-\ell/2}^{\ell/2}\frac{dx'}{(x'^2+y^2)^{3/2}} &= \int_{-\theta}^{\theta}\frac{y\sec^2\theta' d\theta'}{y^3(\tan^2\theta'+1)^{3/2}} = \frac{1}{y^2}\int_{-\theta}^{\theta}\frac{\sec^2\theta' d\theta'}{(\tan^2\theta'+1)^{3/2}} = \frac{1}{y^2}\int_{-\theta}^{\theta}\frac{\sec^2\theta' d\theta'}{\sec\theta'^3} \\ &= \frac{1}{y^2}\int_{-\theta}^{\theta}\frac{d\theta'}{\sec\theta'} = \frac{1}{y^2}\int_{-\theta}^{\theta}\cos\theta'\,d\theta' = \frac{2\sin\theta}{y^2}.\end{aligned} \qquad (2.10.8)$$

Thus the y-component of the electric field is

$$\boxed{E_y = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda\sin\theta}{y} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}\frac{\ell/2}{\sqrt{y^2+(\ell/2)^2}}}. \qquad (2.10.9)$$

In the limit where $y \gg \ell$, the above expression reduces to the "point-charge" limit:

$$E_y \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}\frac{\ell/2}{y} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\ell}{y^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{y^2}. \tag{2.10.10}$$

On the other hand, when $\ell \gg y$, we have

$$E_y \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}. \tag{2.10.11}$$

In this infinite length limit, the system has cylindrical symmetry. In this case, an alternative approach based on Gauss's law can be used to obtain Eq. (2.10.11), as we shall show in Chapter 4. The characteristic behavior of E_y / E_0 (with $E_0 = Q/4\pi\varepsilon_0\ell^2$) as a function of y/ℓ is shown in Figure 2.10.4.

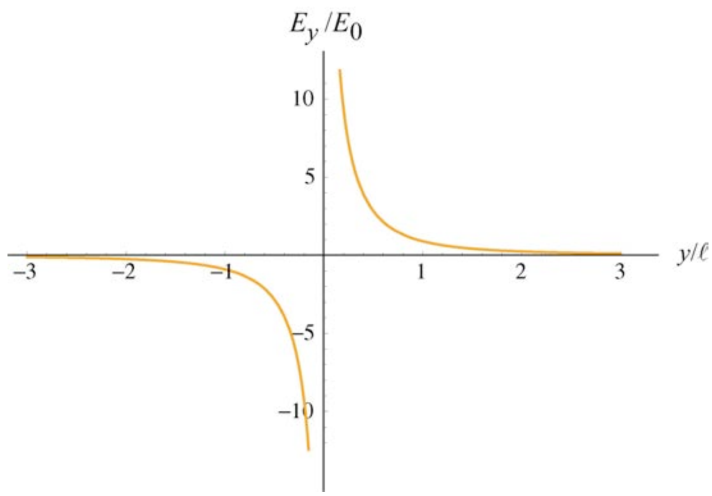


Figure 2.10.4 Electric field of a non-conducting rod as a function of y/ℓ.

Example 2.4: Electric Field on the Axis of a Ring

A non-conducting ring of radius R with a uniform charge density λ and a total charge Q is lying in the xy- plane, as shown in Figure 2.10.5. Compute the electric field at a point P, located at a distance z from the center of the ring along its axis of symmetry.

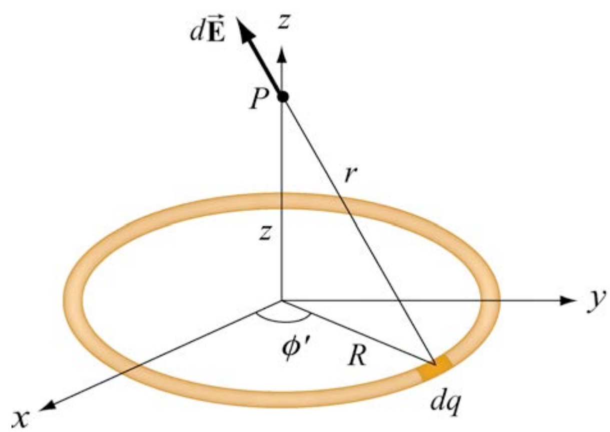


Figure 2.10.5 Electric field at P due to the charge element dq.

Solution:

Consider a small length element $d\ell'$ on the ring.

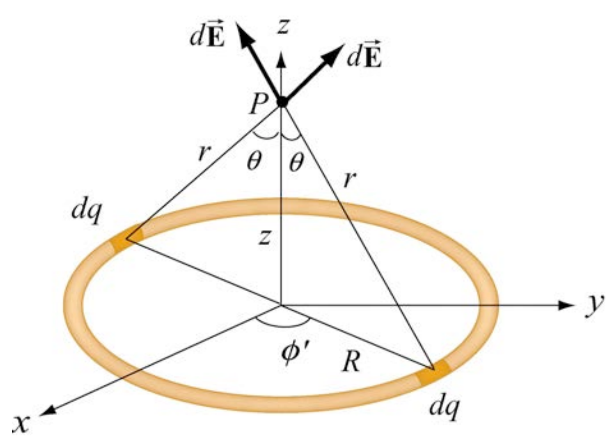


Figure 2.10.6

The amount of charge contained within this element is $dq = \lambda\, d\ell' = \lambda R\, d\phi'$. Its contribution to the electric field at P is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\, d\phi'}{r^2}\hat{\mathbf{r}}. \tag{2.10.12}$$

Using the symmetry argument illustrated in Figure 2.10.6, we see that the electric field at P must point in the $+z$ −direction. The z-component of the electric field is given by

$$dE_z = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\, d\phi'}{R^2+z^2}\frac{z}{\sqrt{R^2+z^2}} = \frac{\lambda}{4\pi\varepsilon_0}\frac{Rz\, d\phi'}{(R^2+z^2)^{3/2}}. \tag{2.10.13}$$

Upon integrating over the entire ring, we obtain

$$E_z = \frac{\lambda}{4\pi\varepsilon_0}\frac{Rz}{(R^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' = \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi Rz}{(R^2+z^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}, \quad (2.10.14)$$

where the total charge is $Q = \lambda(2\pi R)$. A plot of the electric field as a function of z is given in Figure 2.10.7.

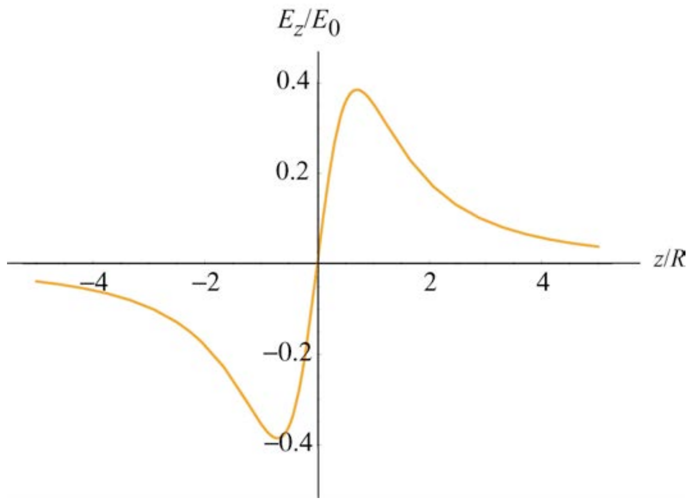


Figure 2.10.7 Electric field along the axis of symmetry of a non-conducting ring of radius R, with $E_0 = Q/4\pi\varepsilon_0 R^2$.

Notice that the electric field at the center of the ring vanishes. This is to be expected from symmetry arguments.

Example 2.5: Electric Field Due to a Uniformly Charged Disk

A uniformly charged disk of radius R with a total charge Q lies in the xy-plane. Find the electric field at a point P, along the z-axis that passes through the center of the disk perpendicular to its plane. What are the limits when $z >> R$ and when $R >> z$?

Solution: By treating the disk as a set of concentric uniformly charged rings, the problem could be solved by using the result obtained in Example 2.4. Consider a ring of radius r' and thickness dr', as shown in Figure 2.10.8.

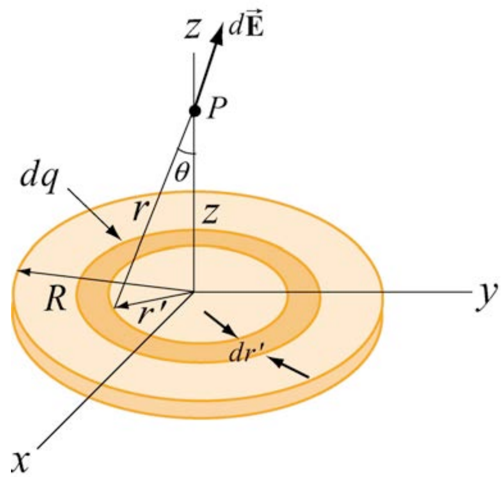


Figure 2.10.8 A uniformly charged disk of radius R.

By symmetry arguments, the electric field at P points in the $+z$-direction. Since the ring has a charge $dq = \sigma(2\pi r'\,dr')$, from Eq. (2.10.14), we see that the ring gives a contribution to the z-component of the electric field

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{z\,dq}{(r'^2+z^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{z(2\pi\sigma r'\,dr')}{(r'^2+z^2)^{3/2}}. \qquad (2.10.15)$$

Integrating from $r' = 0$ to $r' = R$, the z-component of the electric field at P becomes

$$\begin{aligned} E_z &= \int dE_z = \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{r'\,dr'}{(r'^2+z^2)^{3/2}} = \frac{\sigma z}{4\varepsilon_0}\int_{z^2}^{R^2+z^2}\frac{du}{u^{3/2}} = \frac{\sigma z}{4\varepsilon_0}\frac{u^{-1/2}}{(-1/2)}\Bigg|_{z^2}^{R^2+z^2} \\ &= -\frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{z^2}}\right] = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right]. \end{aligned} \qquad (2.10.16)$$

The above equation may be rewritten as

$$E_z = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\left[1 - \dfrac{z}{\sqrt{z^2+R^2}}\right], & z > 0 \\ \dfrac{\sigma}{2\varepsilon_0}\left[-1 - \dfrac{z}{\sqrt{z^2+R^2}}\right], & z < 0 \end{cases}. \qquad (2.10.17)$$

The plot of E_z / E_0 as a function of z / R is shown in Figure 2.10.9, where $E_0 = \sigma / 2\varepsilon_0$.

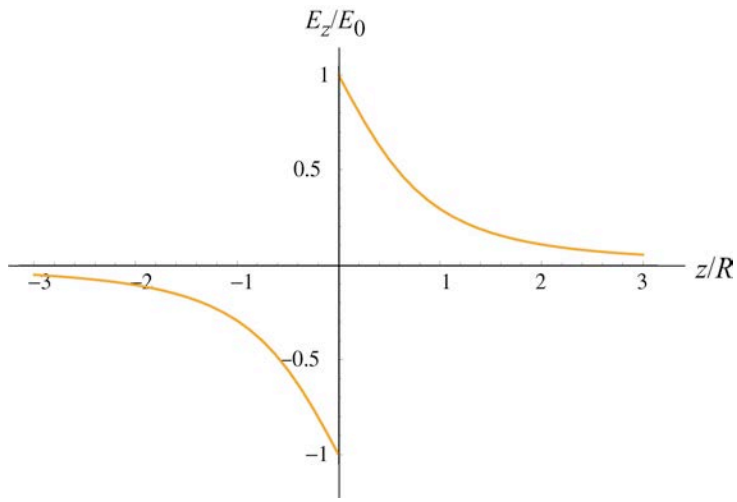


Figure 2.10.9 Electric field of a non-conducting plane of uniform charge density.

To show that the "point-charge" limit is recovered for $z >> R$, we make use of the Taylor-series expansion:

$$1-\frac{z}{\sqrt{z^2+R^2}}=1-\left(1+\frac{R^2}{z^2}\right)^{-1/2}=1-\left(1-\frac{1}{2}\frac{R^2}{z^2}+\cdots\right)\approx\frac{1}{2}\frac{R^2}{z^2}\,. \qquad (2.10.18)$$

The z-component of the electric field is then

$$E_z=\frac{\sigma}{2\varepsilon_0}\frac{R^2}{2z^2}=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\pi R^2}{z^2}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}\,, \qquad (2.10.19)$$

which is indeed the expected "point-charge" result. On the other hand, we may also consider the limit where $R >> z$. Physically this means that the plane is very large, or the field point P is extremely close to the surface of the plane. The electric field in this limit becomes, in unit-vector notation,

$$\vec{\mathbf{E}}=\begin{cases}\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z>0\\[2ex] -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z<0\end{cases}\,. \qquad (2.10.20)$$

The plot of the electric field in this limit is shown in Figure 2.10.10.

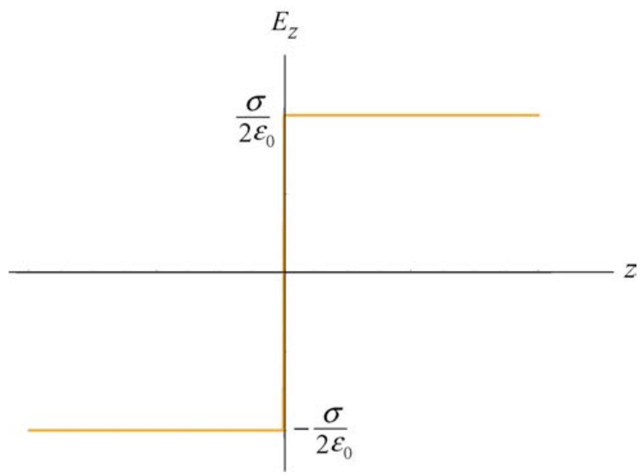


Figure 2.10.10 Electric field of an infinitely large non-conducting plane.

Notice the discontinuity in electric field as we cross the plane. The discontinuity is given by

$$\Delta E_z = E_{z+} - E_{z-} = \frac{\sigma}{2\varepsilon_0} - \left(-\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0}. \quad (2.10.21)$$

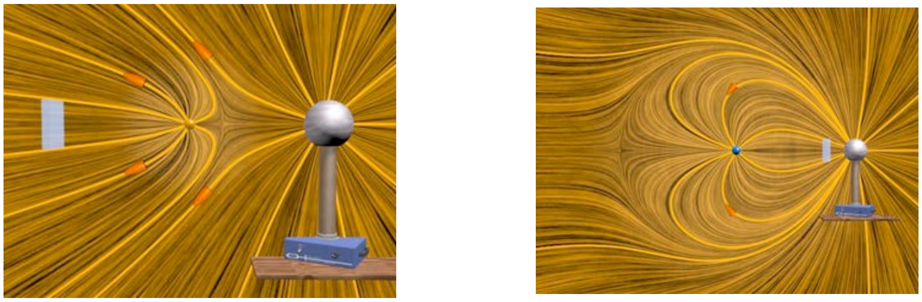
As we shall see in Chapter 4, if a given surface has a charge density σ, then the normal component of the electric field across that surface always exhibits a discontinuity with $\Delta E_n = \sigma / \varepsilon_0$.

2.11 Rubber Bands and Strings and the Forces Transmitted by Electric Fields

We now return to our considerations in Section 1.1.2, where we asserted that depictions of the *total* field, that is the field due to all objects being considered, allows profound insight into the mechanisms whereby fields transmit forces. The stresses transmitted by electromagnetic fields can be understood as analogous to the forces transmitted by rubber bands and strings, but to reach this understanding we must show a representation of the total field, as we do in the four examples following. The examples below show you how Faraday, the father of field theory, understood how his "lines of force" picture explained Coulomb's Law at a more fundamental level than simply stating it, as we did in Equation (2.2.1).

2.11.1 Charge in the Field of a Van de Graaff Movies

Consider Figure 2.11.1(a) below. The figure illustrates the repulsive force transmitted between two objects by their electric fields. The system consists of a charged metal sphere of a van de Graaff generator. This sphere is fixed in space and is not free to move. The other object is a small charged sphere with charge of the same sign, that is free to move (we neglect the force of gravity on this sphere). According to Coulomb's law, these two like charges repel each other. That is, the small sphere experiences a repulsive force away from the van de Graaff sphere.



(a) link3 (b) link4

Figure 2.11.1 (a) Two charges of the same sign that repel one another because of the "stresses" transmitted by electric fields. We use both the "grass seeds" representation and the "field lines" representation of the electric field of the two charges. (b) Two charges of opposite sign that attract one another because of the stresses transmitted by electric fields.

The movie linked to the Figure 2.11.1(a) above depicts the motion of the small sphere and of the electric fields in this situation, where the motion of the electric field in in the direction of $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$, the direction of electromagnetic energy flow. Note that to repeat the motion of the small sphere in the animation, we have the small sphere "bounce off" of a small plastic square fixed in space some distance from the van de Graaff generator.

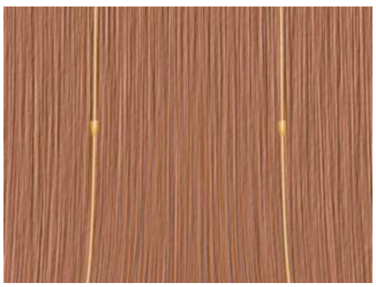
Before we discuss this further, consider Figure 2.11.1(b), which shows one frame of a movie of the interaction of two charges with opposite signs. Here the charge on the small sphere is opposite to that on the van de Graaff sphere. By Coulomb's law, the two objects now attract one another, and the small sphere feels a force attracting it toward the van de Graaff. To repeat the motion of the small sphere in the animation, we have that charge "bounce off" of a plastic square fixed in space near the van de Graaff.

The point of these two movies is to underscore the fact that the Coulomb force between the two charges is *not* "action at a distance." Rather, as we outlined in Section 1.1, the stress is transmitted by direct "contact" from the charges on the van de Graaff to the immediately surrounding space, via the electric field of the charge on the van de Graaff. That stress is then transmitted from one element of space to a neighboring element, in a continuous manner, until it is transmitted to the region of space contiguous to the small sphere, and thus ultimately to the small sphere itself. Although the two spheres are not in direct contact with one another, they are in direct contact with a medium or mechanism that exists between them. The force between the small sphere and the van de Graaff is transmitted (at a finite speed, the speed of light) by stresses induced in the intervening space by the fact that they are charged.

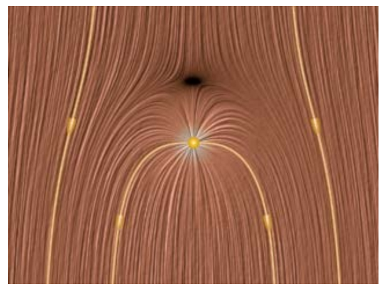
2.11.2 Charged Particle Moving in a Constant Electric Field Movie

As another example of the stresses transmitted by electric fields, and of the interchange of energy between fields and particles, consider a positive electric charge $q > 0$ moving in a constant electric field.

Suppose the charge is initially moving upward along the positive z-axis in a constant background field $\vec{\mathbf{E}} = -E_0\hat{\mathbf{k}}$. Since the charge experiences a constant downward force $\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} = -qE_0\hat{\mathbf{k}}$, it eventually comes to rest (say, at the origin $z = 0$), and then moves back down the negative z-axis. This motion and the fields that accompany it are shown in Figure 2.11.2, at two different times.



(a) link5



(b) link6

Figure 2.11.2 A positive charge moving in a constant electric field which points downward. (a) The total field configuration when the charge is still out of sight on the negative z-axis. (b) The total field configuration when the charge comes to rest at the origin, before it moves back down the negative z-axis.

How do we interpret the motion of the charge in terms of the stresses transmitted by the fields? Faraday would have described the downward force on the charge in Figure 2.11.2(b) as follows. Let the charge be surrounded by an imaginary sphere centered on it, as shown in Figure 2.11.3. The field lines piercing the lower half of the sphere transmit a tension that is parallel to the field. This is a stress pulling downward on the charge from below. The field lines draped over the top of the imaginary sphere transmit a pressure perpendicular to themselves. This is a stress pushing down on the charge from above. The total effect of these stresses is a net downward force on the charge.

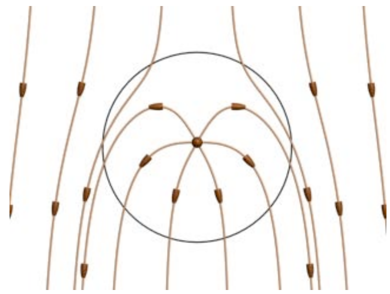


Figure 2.11.3 An electric charge in a constant downward electric field. We enclose the charge in an imaginary sphere in order to discuss the stresses transmitted across the surface of that sphere by the electric field.

Viewing the movie associated with Figure 2.11.2 greatly enhances Faraday's interpretation of the stresses in the static image. As the charge moves upward, it is apparent in the movie that the electric field lines are generally compressed above the charge and stretched below the charge. This field configuration enables the transmission of a downward force to the moving charge we can see as well as an upward force to the charges that produce the constant field, which we cannot see. The overall appearance of the upward motion of the charge through the electric field is that of a point being forced into a resisting medium, with stresses arising in that medium as a result of that encroachment.

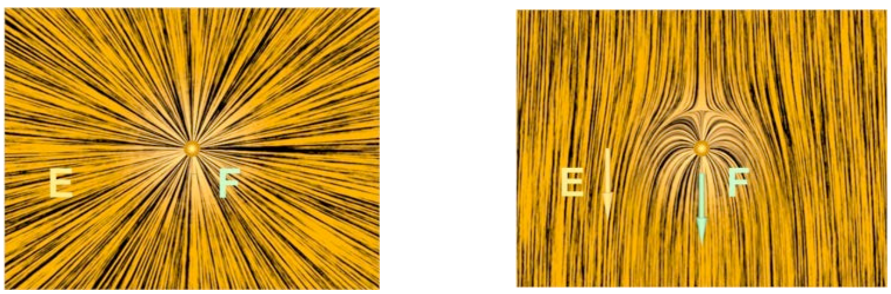
The kinetic energy of the upwardly moving charge is decreasing as more and more energy is stored in the compressed electrostatic field, and conversely when the charge is moving downward. Moreover, because the field line motion in the movie is in the direction of the energy flow, as we discussed in Section 1.8.4, we can explicitly see the electromagnetic energy flow away from the charge into the surrounding field when the charge is slowing. Conversely, we see the electromagnetic energy flow back to the charge from the surrounding field when the charge is being accelerated back down the z-axis by the energy released from the field. The kinetic energy of the charge is flowing into the energy density of the local electric field when it is slowing, and vice versa when it is accelerating.

Finally, consider momentum conservation. The moving charge in the animation of Figure 2.11.2 completely reverses its direction of motion over the course of the animation. How do we conserve momentum in this process? Momentum is conserved because momentum in the positive z-direction is transmitted from the moving charge to the charges that are generating the constant downward electric field (not shown). This is plausible from the field configuration shown in Figure 2.11.2(b). The field stress, which pushes downward on the charge, is accompanied by a stress pushing upward on the charges generating the constant field.

2.11.3 Charged Particle at Rest in a Time-Varying Electric Field Movie

As a third example of the stresses transmitted by electric fields, consider a positive point charge fixed at rest at the origin in an external field that is constant in space but varies in time. This external field is uniform but non-constant and varies according to the equation

$$\vec{\mathbf{E}} = -E_0 \sin^4\left(\frac{2\pi t}{T}\right)\hat{\mathbf{k}}. \tag{2.11.1}$$



(a) link 7 (b) link7

Figure 2.11.4 Two frames of a movie of the electric field around a positive charge fixed at rest in a time-changing electric field that points downward. The orange vector is the electric field and the lighter-colered vector is the force on the point charge.

Figure 2.11.4 shows two frames of a movie of the total electric field configuration for this situation. Figure 2.11.4(a) is at $t = 0$, when the vertical electric field is zero. Frame 2.11.4(b) is at a quarter period later, when the downward electric field is at a maximum. As in Figure 2.11.3 above, we interpret the field configuration in Figure 2.11.4(b) as indicating a net downward force on the stationary charge. The motion of the field lines in the movie associated with Figure 2.11.4, which is in the direction of the electromagnetic energy flow (see Section 1.8.4), shows the dramatic inflow of energy into the neighborhood of the charge as the external electric field grows in time, with a resulting build-up of stress that transmits a downward force to the positive charge.

We can estimate the magnitude of the force on the charge in Figure 2.11.4(b) as follows. At the time shown in Figure 2.11.4(b), the distance r_0 above the charge at which the electric field of the charge is equal and opposite to the constant electric field is determined by the equation

$$E_0 = \frac{q}{4\pi\varepsilon_0 r_0^2}. \tag{2.11.2}$$

The surface area of a sphere of this radius is $A = 4\pi r_0^2 = q/\varepsilon_0 E_0$. In Section 3.4 of Chapter 3, Eq. (3.4.8), we show that the pressure (force per unit area) and/or tension transmitted across the surface of this sphere surrounding the charge is of the order of

$\varepsilon_0 E^2 / 2$. Because the electric field on the surface of the sphere is of order E_0, the total force transmitted by the field is of order $\varepsilon_0 {E_0}^2 / 2$ times the area of the sphere, or

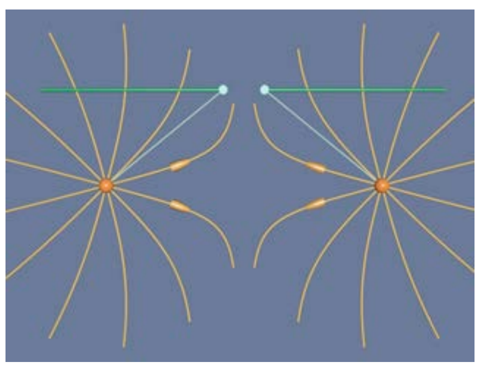
$$(\varepsilon_0 {E_0}^2 / 2)(4\pi r_0^2) = (\varepsilon_0 {E_0}^2 / 2)(q / \varepsilon_0 E_0) \approx qE_0 \,,$$

as we expect.

Of course this downward net force is a combination of a pressure pushing down on the top of the sphere and a tension pulling down across the bottom of the sphere. However, the rough estimate that we have just made demonstrates that the pressures and tensions transmitted across the surface of this sphere surrounding the charge are plausibly of order $\varepsilon_0 E^2 / 2$.

2.11.4 Like and Unlike Charges Hanging from Pendulums Movies

Consider two charges hanging from pendulums whose supports can be moved closer or further apart by an external agent. First, suppose the charges both have the same sign, and therefore repel.



link8

Figure 2.11.5 Two pendulums from which are suspended charges of the same sign. We artificially terminate the field lines at a fixed distance from the charges to avoid visual confusion

Figure 2.11.5 shows the situation when an external agent tries to move the supports (from which the two positive charges are suspended) together. The force of gravity is pulling the charges down, and the force of electrostatic repulsion is pushing them apart on the radial line joining them. The behavior of the electric fields in this situation is an example of an electrostatic pressure transmitted perpendicular to the field. That pressure tries to keep the two charges apart in this situation, as the external agent controlling the pendulum supports tries to move them together. When we move the supports together the charges are pushed apart by the pressure transmitted perpendicular to the electric field.

In contrast, suppose the charges are of opposite signs, and therefore attract. Figure 2.11.6 shows the situation when an external agent moves the supports (from which the two charges are suspended) together. The force of gravity is pulling the charges down, and the force of electrostatic attraction is pulling them together on the radial line joining them. The behavior of the electric fields in this situation is an example of the tension transmitted parallel to the field. That tension tries to pull the two unlike charges together in this situation.

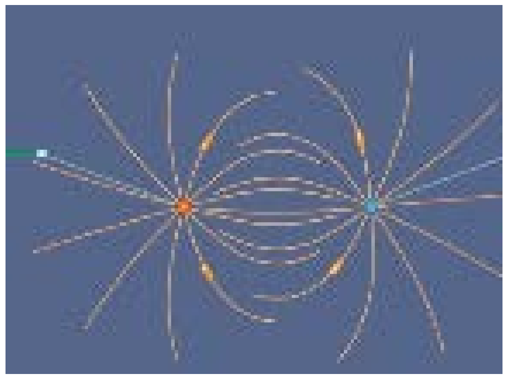
 link9

Figure 2.11.6 Two pendulums with suspended charges of opposite sign.

When we move the supports together the unlike charges are pulled together by the tension transmitted parallel to the electric field. We artificially terminate the field lines at a fixed distance from the charges to avoid visual confusion.

2.11.5 Pressures and Tensions Transmitted by Electric Fields

Let us move from these specific examples to a more general discussion of how Faraday understood the forces transmitted by fields (see also Section 1.1). To do this, we consider a more general case where a closed surface (an imaginary box) is placed in an electric field, as shown in Figure 2.11.7.

If we look at the top face of the imaginary box, there is an electric field pointing in the outward normal direction of that face. From Faraday's field theory perspective, we would say that the field on that face transmits a tension along itself across the face, thereby resulting in an *upward pull*, just as if we had attached a string under tension to that face to pull it upward. Similarly, if we look at the bottom face of the imaginary box, the field on that face is anti-parallel to the outward normal of the face, and according to Faraday's interpretation, we would again say that the field on the bottom face transmits a tension along itself, giving rise to a *downward pull*, just as if a string has been attached to that face to pull it downward. Note that this is a *pull* parallel to the outward normal of the bottom face, regardless of whether the field is into the surface or out of the surface.

If we want to know the total electric force transmitted to the interior of this imaginary box in the up-down direction, we add these two transmitted stresses. If the electric field is homogeneous, the total force transmitted to the interior of the box in the up-down direction is a pull upward plus an equal and opposite pull downward, and adds to zero.

In contrast, if the top of this imaginary box is sitting inside a capacitor, for which the electric field is vertical and constant, and the bottom is sitting inside one of the plates of the capacitor, where the electric field is zero, then there is a net pull upward, and we say that the electric field exerts a upward tension on the plates of the capacitor. We can deduce this by simply looking at the shape of the total electric field. A quantitative calculation of the tension transmitted by electric fields in this example is presented in Section 3.4, see Eq. (3.4.8).

For the left side of the imaginary box, the field on that face is perpendicular to the outward normal of that face, and Faraday would have said that the field on that face transmits a pressure perpendicular to itself, causing a *push* to the *right*. Similarly, for the right side of the imaginary box, the field on that face is perpendicular to the outward normal of the face, and the field would transmit a pressure perpendicular to itself. In this case, there is a *push* to the *left*.

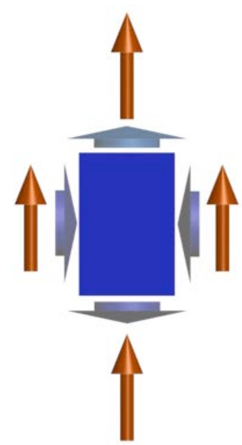


Figure 2.11.7 An imaginary blue box in an electric field (long orange vectors). The short gray vectors indicate the directions of stresses transmitted across the surface of the imaginary box by the field, either pressures (on the left or right faces of the box) or tensions (on the top and bottom faces of the box).

Note that the term "tension" is used when the stress transmitted by the field is parallel (or anti-parallel) to the outward normal of the surface, and "pressure" when it is perpendicular to the outward normal. The magnitude of these pressures and tensions on the various faces of the imaginary surface in Figure 2.11.7 is given by $\varepsilon_0 E^2 / 2$ for the electric field. This quantity has units of force per unit area, or pressure. It is also the energy density stored in the electric field since energy per unit volume has the same units as pressure (see Section 5.3).

2.12 Summary

- The electric force exerted by a charge q_1 on a second charge q_2 is given by **Coulomb's law**:

$$\vec{\mathbf{F}}_{12} = k_e \frac{q_1 q_2}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}} ,$$

where

$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.99\times10^9 \ \mathrm{N\cdot m^2/C^2}$$

is the Coulomb constant.

- The **electric field** at a point in space is defined as the electric force acting on a test charge q_0 divided by q_0:

$$\vec{\mathbf{E}} = \lim_{q_0\to 0}\frac{\vec{\mathbf{F}}_e}{q_0} .$$

- The electric field at a distance r from a charge q is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} .$$

- Using the **superposition principle**, the electric field due to a collection of point charges, each having charge q_i and located at a distance r_i away is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{\mathbf{r}}_i .$$

- A particle of mass m and charge q moving in an electric field $\vec{\mathbf{E}}$ has an acceleration

$$\vec{\mathbf{a}} = \frac{q\vec{\mathbf{E}}}{m} .$$

- An **electric dipole** consists of two equal but opposite charges. The electric dipole moment vector $\vec{\mathbf{p}}$ points from the negative charge to the positive charge, and has a magnitude

$$p = 2aq .$$

- The **torque** acting on an electric dipole places in a uniform electric field $\vec{\mathbf{E}}$ is

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}} .$$

- The **potential energy** of an electric dipole in a uniform external electric field $\vec{\mathbf{E}}$ is

$$U = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}} .$$

- The electric field at a point in space due to a continuous charge element dq is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} .$$

- At sufficiently far away from a continuous charge distribution of finite extent, the electric field approaches the "point-charge" limit.

- As discussed in Section 1.1.2, if we look at the shape of electric field lines for the total electric field, as Faraday did, the electric forces transmitted by fields can be understood at a more fundamental level by analogy to the more familiar forces exerted by strings and rubber bands.

2.13 Problem-Solving Strategies

In this chapter, we have discussed how electric field can be calculated for both the discrete and continuous charge distributions. For the former, we apply the superposition principle:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{\mathbf{r}}_i .$$

For the latter, we must evaluate the vector integral

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{\mathbf{r}} ,$$

where r is the distance from dq to the field point P and $\hat{\mathbf{r}}$ is the corresponding unit vector. To complete the integration, we shall follow the procedures outlined below:

(1) Start with $d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}}$.

(2) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda\, d\ell & \text{(length)} \\ \sigma\, dA & \text{(area)} \\ \rho\, dV & \text{(volume)} \end{cases},$$

depending on whether the charge is distributed over a length, an area, or a volume.

(3) Substitute dq into the expression for $d\vec{\mathbf{E}}$.

(4) Specify an appropriate coordinate system (Cartesian, cylindrical or spherical) and express the differential element ($d\ell$, dA, or dV) and r in terms of the coordinates (see Table 2.1 below for summary.)

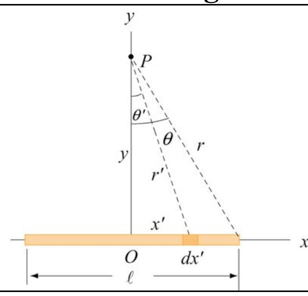
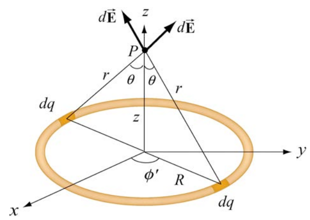
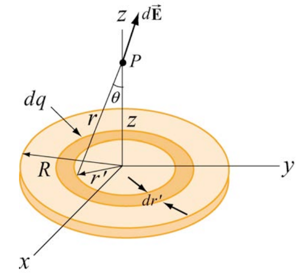
	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
dl	$dx,\ dy,\ dz$	$d\rho,\ \rho d\phi,\ dz$	$dr,\ r\,d\theta,\ r\sin\theta\, d\phi$
dA	$dx\,dy,\ dy\,dz,\ dz\,dx$	$d\rho\, dz,\ \rho\, d\phi\, dz,\ \rho\, d\phi\, d\rho$	$r\,dr\,d\theta,\ r\sin\theta\, dr\, d\phi,\ r^2\sin\theta\, d\theta\, d\phi$
dV	$dx\,dy\,dz$	$\rho\, d\rho d\phi\, dz$	$r^2\sin\theta\, dr\, d\theta\, d\phi$

Table 2.1 Differential elements of length, area and volume in different coordinates

(5) Rewrite $d\vec{\mathbf{E}}$ in terms of the integration variable(s), and apply symmetry argument to identify non-vanishing component(s) of the electric field.

(6) Complete the integration to obtain $\vec{\mathbf{E}}$.

In the Table below we illustrate how the above methodologies can be utilized to compute the electric field for an infinite line charge, a ring of charge and a uniformly charged disk.

	Line charge	Ring of charge	Uniformly charged disk
Figure			
(2) Express dq in terms of charge density	$dq = \lambda\, dx'$	$dq = \lambda\, d\ell$	$dq = \sigma\, dA$
(3) Write down dE	$dE = k_e \dfrac{\lambda\, dx'}{r'^2}$	$dE = k_e \dfrac{\lambda\, dl}{r^2}$	$dE = k_e \dfrac{\sigma\, dA}{r^2}$
(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $\cos\theta = \dfrac{y}{r'}$ $r' = \sqrt{x'^2 + y^2}$	$d\ell = R\, d\phi'$ $\cos\theta = \dfrac{z}{r}$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r' dr'$ $\cos\theta = \dfrac{z}{r}$ $r = \sqrt{r'^2 + z^2}$
(5) Apply symmetry argument to identify non-vanishing component(s) of dE	$dE_y = dE\cos\theta = k_e \dfrac{\lambda y dx'}{(x'^2 + y^2)^{3/2}}$	$dE_z = dE\cos\theta = k_e \dfrac{\lambda R z\, d\phi'}{(R^2 + z^2)^{3/2}}$	$dE_z = dE\cos\theta = k_e \dfrac{2\pi\sigma z r'\, dr'}{(r'^2 + z^2)^{3/2}}$
(6) Integrate to get E	$E_y = k_e \lambda y \int_{-\ell/2}^{+\ell/2} \dfrac{dx}{(x^2 + y^2)^{3/2}} = \dfrac{2k_e\lambda}{y} \dfrac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = k_e \dfrac{R\lambda z}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi' = k_e \dfrac{(2\pi R\lambda) z}{(R^2 + z^2)^{3/2}} = k_e \dfrac{Qz}{(R^2 + z^2)^{3/2}}$	$E_z = 2\pi\sigma k_e z \int_0^R \dfrac{r'\, dr'}{(r'^2 + z^2)^{3/2}} = 2\pi\sigma k_e \left(\dfrac{z}{\lvert z \rvert} - \dfrac{z}{\sqrt{z^2 + R^2}} \right)$

2.14 Solved Problems

2.14.1 Hydrogen Atom

In the classical model of the hydrogen atom, the electron revolves around the proton with a radius of $r = 5.3 \times 10^{-11}\,\mathrm{m}$. The magnitude of the charge of the electron and proton is $e = 1.6 \times 10^{-19}\,\mathrm{C}$.

(a) What is the magnitude of the electric force between the proton and the electron?

(b) What is the magnitude of the electric field due to the proton at r ?

(c) What is ratio of the magnitudes of the electrical and gravitational force between electron and proton? Does the result depend on the distance between the proton and the electron? The mass of the electron is $m_e = 9.1\times10^{-31}\,\text{kg}$ and the mass of the proton is $m_p = 1.7\times10^{-27}\,\text{kg}$.

(d) In light of your calculation in (b), explain why electrical forces do not influence the motion of planets.

Solutions:

(a) The magnitude of the force is given by

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}.$$

Now we can substitute our numerical values and find that the magnitude of the force between the proton and the electron in the hydrogen atom is

$$F_e = \frac{(9.0\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2)(1.6\times10^{-19}\,\text{C})^2}{(5.3\times10^{-11}\,\text{m})^2} = 8.2\times10^{-8}\,\text{N}.$$

(b) The magnitude of the electric field due to the proton is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{(9.0\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2)(1.6\times10^{-19}\,\text{C})}{(0.5\times10^{-10}\,\text{m})^2} = 5.76\times10^{11}\,\text{N}/\text{C}.$$

(c) The ratio of the magnitudes of the electric and gravitational force is given by

$$\gamma = \frac{\left(\dfrac{1}{4\pi\varepsilon_0}\dfrac{e^2}{r^2}\right)}{\left(G\dfrac{m_p m_e}{r^2}\right)} = \frac{\dfrac{1}{4\pi\varepsilon_0}e^2}{G m_p m_e} =$$

$$= \frac{(9.0\times10^{9}\,\mathrm{N\cdot m^2/C^2})(1.6\times10^{-19}\,\mathrm{C})^2}{(6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2})(1.7\times10^{-27}\,\mathrm{kg})(9.1\times10^{-31}\,\mathrm{kg})} = 2.2\times10^{39}$$

,

This ratio is independent of r, the distance between the proton and the electron.

(d) The electric force is 39 orders of magnitude stronger than the gravitational force between the electron and the proton. Why do the gravitational forces and not the electrical forces determine the large-scale motions of planets? The answer is that the magnitudes of the charge of the electron and proton are equal. The best experiments show that the difference between these magnitudes is a number on the order of 10^{-24}. Since objects like planets have essentially the same number of protons as electrons, they are electrically neutral. Therefore the force between planets is entirely determined by gravity.

2.14.2 Millikan Oil-Drop Experiment

An oil drop of radius $R = 1.64\times10^{-6}\,\mathrm{m}$ and mass density $\rho_{\mathrm{oil}} = 8.51\times10^{2}\,\mathrm{kg/m^3}$ is allowed to fall from rest and then enters into a region of constant external field $\vec{\mathbf{E}}$ applied in the downward direction. The oil drop has an unknown electric charge q (due to irradiation by bursts of X-rays). The magnitude of the electric field is adjusted until the gravitational force $\vec{\mathbf{F}}_g$ on the oil drop is exactly balanced by the electric force, $\vec{\mathbf{F}}_e$. Suppose this balancing occurs when the electric field is $\vec{\mathbf{E}} = -E_y\,\hat{\mathbf{j}} = -(1.92\times10^{5}\ \mathrm{N/C})\,\hat{\mathbf{j}}$, with $E_y = 1.92\times10^{5}\ \mathrm{N/C}$.

(a) What is the mass of the oil drop?

(b) What is the charge Q on the oil drop in units of electronic charge $e = 1.6\times10^{-19}\,\mathrm{C}$?

Solutions:

(a) Assume that the oil drop is a sphere of radius R with volume $V = (4/3)\pi R^3$. Then the mass M of the oil drop is

$$M = \rho_{\mathrm{oil}} V = \rho_{\mathrm{oil}}(4/3)\pi R^3 .$$

Now we can substitute our numerical values into our symbolic expression for the mass,

$$M=\rho_{\text{oil}}(4\pi/3)R^3=(8.51\times10^2\ \text{kg}/\text{m}^3)(4\pi/3)(1.64\times10^{-6}\ \text{m})^3=1.57\times10^{-14}\,\text{kg}\,.$$

(b) The oil drop will be in static equilibrium when the gravitational force exactly balances the electrical force: $\vec{\mathbf{F}}_g+\vec{\mathbf{F}}_e=\vec{\mathbf{0}}$. Since the gravitational force points downward, the electric force on the oil must be upward. Using our force laws, we have

$$0=M\vec{\mathbf{g}}+Q\vec{\mathbf{E}}\quad\Rightarrow\quad Mg=-QE_y\,.$$

With the electrical field pointing downward, we conclude that the charge on the oil drop must be negative. Notice that we have chosen the unit vector $\hat{\mathbf{j}}$ to point upward. We can solve this equation for the charge on the oil drop:

$$Q=-\frac{Mg}{E_y}=-\frac{(1.57\times10^{-14}\text{kg})(9.80\,\text{m}/\text{s}^2)}{1.92\times10^5\ \text{N}/\text{C}}=-8.03\times10^{-19}\text{C}\,.$$

Since the electron has charge $e=1.6\times10^{-19}\,\text{C}$, the charge of the oil drop in units of e is

$$N=\frac{Q}{e}=\frac{8.02\times10^{-19}\,\text{C}}{1.6\times10^{-19}\,\text{C}}=5\,.$$

You may at first be surprised that this number is an integer, but the Millikan oil drop experiment was the first direct experimental evidence that charge is quantized. Thus, from the given data we can assert that there are five electrons on the oil drop!

2.14.3 Charge Moving Perpendicularly to an Electric Field

An electron is injected horizontally into a uniform field produced by two oppositely charged plates, as shown in Figure 2.14.1. The particle has an initial velocity $\vec{\mathbf{v}}_0=v_0\,\hat{\mathbf{i}}$ perpendicular to $\vec{\mathbf{E}}$.

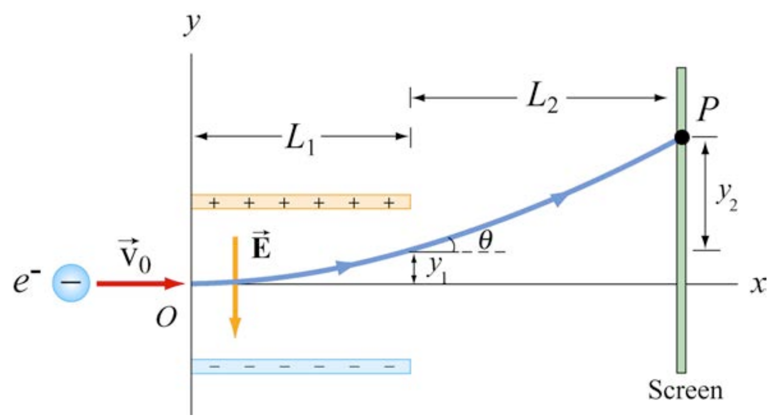


Figure 2.14.1 Charge moving perpendicular to an electric field

(a) While between the plates, what is the force on the electron?

(b) What is the acceleration of the electron when it is between the plates?

(c) The plates have length L_1 in the x-direction. At what time t_1 will the electron leave the plate?

(d) Suppose the electron enters the electric field at time $t = 0$. What is the velocity of the electron at time t_1 when it leaves the plates?

(e) What is the vertical displacement of the electron after time t_1 when it leaves the plates?

(f) What angle θ_1 does the electron make with the horizontal, when the electron leaves the plates at time t_1?

(g) The electron hits the screen located a distance L_2 from the end of the plates at a time t_2. What is the total vertical displacement of the electron from time $t = 0$ until it hits the screen at t_2?

Solutions:

(a) Since the electron has a negative charge, $q = -e$, the force on the electron is

$$\vec{\mathbf{F}}_e = q\vec{\mathbf{E}} = -e\vec{\mathbf{E}} = (-e)(-E_y)\hat{\mathbf{j}} = eE_y\,\hat{\mathbf{j}}\,.$$

where the electric field is written as $\vec{\mathbf{E}} = -E_y\,\hat{\mathbf{j}}$, with $E_y > 0$. The force on the electron is upward. Note that the motion of the electron is analogous to the motion of a mass that is thrown horizontally in a uniform gravitation field. The mass follows a parabolic trajectory downward. Since the electron is negatively charged, the constant force on the electron is upward and the electron will be deflected upwards on a parabolic path.

(b) The acceleration of the electron is

$$\vec{\mathbf{a}} = \frac{q\vec{\mathbf{E}}}{m} = -\frac{qE_y}{m}\hat{\mathbf{j}} = \frac{eE_y}{m}\hat{\mathbf{j}},$$

and its direction is upward.

(c) The time of passage for the electron is given by $t_1 = L_1 / v_0$. The time t_1 is not affected by the acceleration because v_0, the horizontal component of the velocity which determines the time, is not affected by the field.

(d) The electron has an initial horizontal velocity, $\vec{\mathbf{v}}_\mathbf{0} = v_0\,\hat{\mathbf{i}}$. Since the acceleration of the electron is in the positive y-direction, only the y-component of the velocity changes. The velocity at a later time t_1 is given by

$$\vec{\mathbf{v}} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} = v_0\,\hat{\mathbf{i}} + a_y\,t_1\,\hat{\mathbf{j}} = v_0\,\hat{\mathbf{i}} + \left(\frac{eE_y}{m}\right)t_1\,\hat{\mathbf{j}} = v_0\,\hat{\mathbf{i}} + \left(\frac{eE_yL_1}{mv_0}\right)\hat{\mathbf{j}}.$$

(e) From the figure, we see that the electron travels a horizontal distance L_1 in the time $t_1 = L_1 / v_0$ and then emerges from the plates with a vertical displacement

$$y_1 = \frac{1}{2}a_y\,t_1^{\,2} = \frac{1}{2}\left(\frac{eE_y}{m}\right)\left(\frac{L_1}{v_0}\right)^2.$$

(f) When the electron leaves the plates at time t_1, the electron makes an angle θ_1 with the horizontal given by the ratio of the components of its velocity,

$$\tan\theta = \frac{v_y}{v_x} = \frac{(eE_y / m)(L_1 / v_0)}{v_0} = \frac{eE_y\,L_1}{mv_0^{\,2}}.$$

(g) After the electron leaves the plate, there is no longer any force on the electron so it travels in a straight path. The deflection y_2 is

$$y_2 = L_2 \tan\theta_1 = \frac{eE_y L_1 L_2}{m v_0^2},$$

and the total deflection becomes

$$\boxed{y = y_1 + y_2 = \frac{1}{2}\frac{eE_y L_1^2}{m v_0^2} + \frac{eE_y L_1 L_2}{m v_0^2} = \frac{eE_y L_1}{m v_0^2}\left(\frac{1}{2}L_1 + L_2\right)}.$$

2.14.4 Electric Field of a Dipole

Consider the electric dipole moment shown in Figure 2.7.1.

(a) Show that the electric field of the dipole in the limit where $r >> a$ is

$$E_x = \frac{3p}{4\pi\varepsilon_0 r^3}\sin\theta\cos\theta, \quad E_y = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)$$

where $\sin\theta = x/r$ and $\cos\theta = y/r$.

(b) Show that the above expression for the electric field can also be written in terms of the polar coordinates as

$$\vec{\mathbf{E}}(r,\theta) = E_r\,\hat{\mathbf{r}} + E_\theta\,\hat{\boldsymbol{\theta}}$$

where

$$\boxed{E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}, \quad E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}}.$$

Solutions:

(a) Let's compute the electric field strength at a distance $r >> a$ due to the dipole. The x-component of the electric field strength at the point P with Cartesian coordinates $(x, y, 0)$ is given by

$$E_x = \frac{q}{4\pi\varepsilon_0}\left(\frac{\cos\theta_+}{r_+^2} - \frac{\cos\theta_-}{r_-^2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{x}{\left[x^2 + (y-a)^2\right]^{3/2}} - \frac{x}{\left[x^2 + (y+a)^2\right]^{3/2}}\right),$$

where

$$r_\pm^2 = r^2 + a^2 \mp 2ra\cos\theta = x^2 + (y \mp a)^2.$$

Similarly, the y-component is given by

$$E_y = \frac{q}{4\pi\varepsilon_0}\left(\frac{\sin\theta_+}{r_+^{\,2}} - \frac{\sin\theta_-}{r_-^{\,2}}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{y-a}{\left[x^2+(y-a)^2\right]^{3/2}} - \frac{y+a}{\left[x^2+(y+a)^2\right]^{3/2}}\right).$$

We shall make a polynomial expansion for the electric field using the Taylor-series expansion. We will then collect terms that are proportional to $1/r^3$ and ignore terms that are proportional to $1/r^5$, where $r = (x^2+y^2)^{1/2}$. We begin with

$$(x^2+(y\pm a)^2)^{-3/2} = (x^2+y^2+a^2\pm 2ay)^{-3/2} = r^{-3}\left(1+\frac{a^2\pm 2ay}{r^2}\right)^{-3/2}.$$

In the limit where $r \gg a$, we use the Taylor-series expansion with $s = (a^2 \pm 2ay)/r^2$:

$$(1+s)^{-3/2} = 1 - \frac{3}{2}s + \frac{15}{8}s^2 - \cdots.$$

The above equations for the components of the electric field becomes

$$E_x = \frac{q}{4\pi\varepsilon_0}\frac{6xya}{r^5} + \cdots$$

and

$$E_y = \frac{q}{4\pi\varepsilon_0}\left(-\frac{2a}{r^3} + \frac{6y^2a}{r^5}\right) + \cdots,$$

where we have neglected the terms of order equal to and greater than s^2 (all the terms of order s^2 and higher are denoted by the symbol $O(s^2)$). The electric field can then be written as

$$\vec{\mathbf{E}} = E_x\,\hat{\mathbf{i}} + E_y\,\hat{\mathbf{j}} = \frac{q}{4\pi\varepsilon_0}\left[-\frac{2a}{r^3}\hat{\mathbf{j}} + \frac{6ya}{r^5}(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}})\right] = \frac{p}{4\pi\varepsilon_0 r^3}\left[\frac{3yx}{r^2}\hat{\mathbf{i}} + \left(\frac{3y^2}{r^2} - 1\right)\hat{\mathbf{j}}\right],$$

where we have made used of the definition of the magnitude of the electric dipole moment $p = 2aq$.

In terms of polar coordinates, with $\sin\theta = x/r$ and $\cos\theta = y/r$, (as seen from Figure 2.14.4), we obtain the desired results:

$$\boxed{E_x = \frac{3p}{4\pi\varepsilon_0 r^3}\sin\theta\cos\theta, \quad E_y = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)}.$$

(b) We begin with the expression obtained in (a) for the electric dipole in Cartesian coordinates:

$$\vec{\mathbf{E}}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left[3\sin\theta\cos\theta\,\hat{\mathbf{i}} + (3\cos^2\theta - 1)\,\hat{\mathbf{j}}\right].$$

With a little algebra, the above expression may be rewritten as

$$\begin{aligned}\vec{\mathbf{E}}(r,\theta) &= \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) + \sin\theta\cos\theta\,\hat{\mathbf{i}} + (\cos^2\theta - 1)\,\hat{\mathbf{j}}\right] \\ &= \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) + \sin\theta(\cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}})\right],\end{aligned}$$

where we used the trigonometric identity $(\cos^2\theta - 1) = -\sin^2\theta$. Because the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in polar coordinates can be decomposed as

$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}},\end{aligned}$$

the electric field in polar coordinates is given by

$$\boxed{\vec{\mathbf{E}}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right]}.$$

The magnitude of $\vec{\mathbf{E}}$ is

$$E = (E_r^2 + E_\theta^2)^{1/2} = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta + 1)^{1/2}.$$

2.14.5 Electric Field of an Arc

A thin rod with a uniform charge per unit length λ is bent into the shape of an arc of a circle of radius R. The arc subtends a total angle $2\theta_0$, symmetric about the x-axis, as shown in Figure 2.14.2. What is the electric field $\vec{\mathbf{E}}$ at the origin O?

Solution: Consider a differential element of length $d\ell = R\,d\theta$, which makes an angle θ with the x-axis, as shown in Figure 2.14.2(b). The amount of charge it carries is $dq = \lambda\, d\ell = \lambda R\, d\theta$. The contribution to the electric field at O is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}(-\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}}) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, d\theta}{R}(-\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}}) .$$

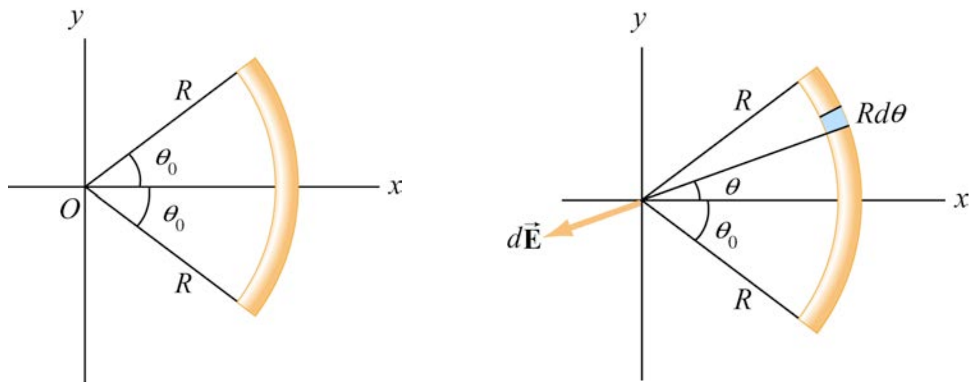


Figure 2.14.2 (a) Geometry of charged source. (b) Charge element dq

Integrating over the angle from $-\theta_0$ to $+\theta_0$, we have

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{R}\int_{-\theta_0}^{\theta_0} d\theta(-\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}}) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{R}(-\cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}})\Big|_{-\theta_0}^{\theta_0} = -\frac{1}{4\pi\varepsilon_0}\frac{2\lambda\sin\theta_0}{R}\hat{\mathbf{i}} .$$

We see that the electric field only has the x-component, as required by a symmetry argument. If we take the limit $\theta_0 \to \pi$, the arc becomes a circular ring. Since $\sin\pi = 0$, the equation above implies that the electric field at the center of a non-conducting ring is zero. This is to be expected from symmetry arguments. On the other hand, for very small θ_0, $\sin\theta_0 \approx \theta_0$ and we recover the point-charge limit:

$$\vec{\mathbf{E}} \approx -\frac{1}{4\pi\varepsilon_0}\frac{2\lambda\theta_0}{R}\hat{\mathbf{i}} = -\frac{1}{4\pi\varepsilon_0}\frac{2\lambda\theta_0 R}{R^2}\hat{\mathbf{i}} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\hat{\mathbf{i}} ,$$

where the charge on the arc is $Q = \lambda\ell = \lambda(2R\theta_0)$.

2.14.6 Electric Field Off the Axis of a Finite Rod

A non-conducting rod of length ℓ with a uniform charge density λ and charge Q is lying along the x-axis, as illustrated in Figure 2.14.3. Compute the electric field at a point P, located at a distance y off the axis of the rod.

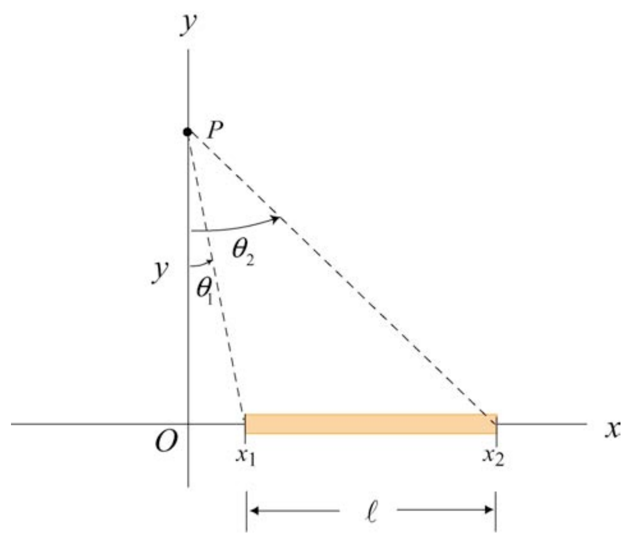


Figure 2.14.3

Solution: The problem can be solved by following the procedure used in Example 2.3. Consider a length element dx' on the rod, as shown in Figure 2.13.4. The charge carried by the element is $dq = \lambda\, dx'$.

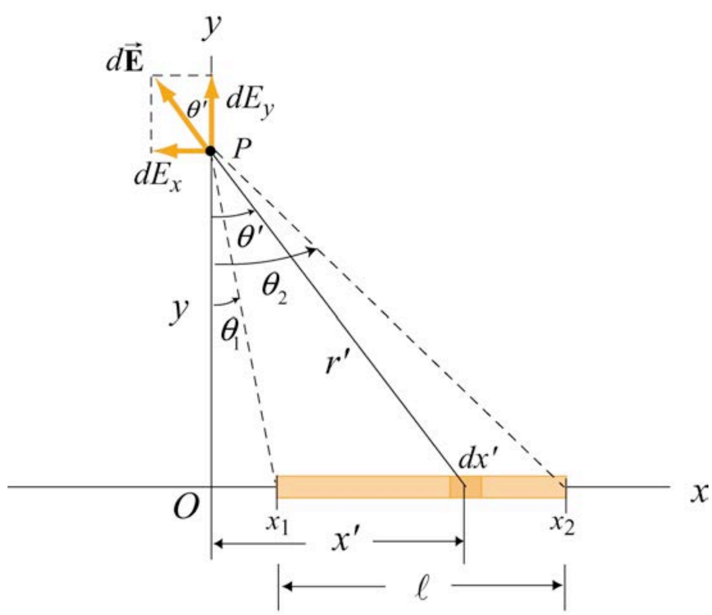


Figure 2.14.4

The electric field at P produced by this element is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r'^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2 + y^2}(-\sin\theta'\,\hat{\mathbf{i}} + \cos\theta'\,\hat{\mathbf{j}}),$$

where the unit vector $\hat{\mathbf{r}}$ has been written in Cartesian coordinates: $\hat{\mathbf{r}} = -\sin\theta'\,\hat{\mathbf{i}} + \cos\theta'\,\hat{\mathbf{j}}$. In the absence of symmetry, the field at P has both the x- and y-components. The x-component of the electric field is

$$dE_x = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2+y^2}\sin\theta' = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2+y^2}\frac{x'}{\sqrt{x'^2+y^2}} = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda x'\, dx'}{(x'^2+y^2)^{3/2}}.$$

Integrating from $x' = x_1$ to $x' = x_2$, we have

$$\begin{aligned} E_x &= -\frac{\lambda}{4\pi\varepsilon_0}\int_{x_1}^{x_2}\frac{x'\,dx'}{(x'^2+y^2)^{3/2}} = -\frac{\lambda}{4\pi\varepsilon_0}\frac{1}{2}\int_{x_1^2+y^2}^{x_2^2+y^2}\frac{du}{u^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0}u^{-1/2}\Bigg|_{x_1^2+y^2}^{x_2^2+y^2} \\ &= \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x_2^2+y^2}} - \frac{1}{\sqrt{x_1^2+y^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0 y}\left[\frac{y}{\sqrt{x_2^2+y^2}} - \frac{y}{\sqrt{x_1^2+y^2}}\right] \\ &= \frac{\lambda}{4\pi\varepsilon_0 y}(\cos\theta_2 - \cos\theta_1). \end{aligned}$$

Similarly, the y-component of the electric field due to the charge element is

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2+y^2}\cos\theta' = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2+y^2}\frac{y}{\sqrt{x'^2+y^2}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda y dx'}{(x'^2+y^2)^{3/2}}.$$

Integrating over the entire length of the rod, we obtain

$$E_y = \frac{\lambda y}{4\pi\varepsilon_0}\int_{x_1}^{x_2}\frac{dx'}{(x'^2+y^2)^{3/2}} = \frac{\lambda y}{4\pi\varepsilon_0}\frac{1}{y^2}\int_{\theta_1}^{\theta_2}\cos\theta'\,d\theta' = \frac{\lambda}{4\pi\varepsilon_0 y}(\sin\theta_2 - \sin\theta_1).$$

where we have used the result obtained in Eq. (2.10.8) in completing the integration.

In the infinite length limit where $x_1 \to -\infty$ and $x_2 \to +\infty$, with $x_i = y\tan\theta_i$, the corresponding angles are $\theta_1 = -\pi/2$ and $\theta_2 = +\pi/2$. Substituting the values into the expressions above, we have

$$E_x = 0, \qquad E_y = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y},$$

in agreement with the result shown in Eq. (2.10.11).

2.15 Conceptual Questions

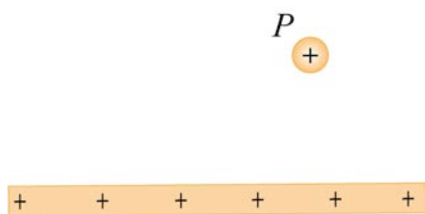
1. Compare and contrast Newton's law of gravitation, $\vec{\mathbf{F}}_G = -(Gm_1m_2 / r^2)\hat{\mathbf{r}}$, and Coulomb's law, $\vec{\mathbf{F}}_e = (k_e q_1 q_2 / r^2)\hat{\mathbf{r}}$.

2. Can electric field lines cross each other? Explain.

3. Two opposite charged objects are placed on a line as shown in the figure below.

The charge on the right is three times the magnitude of the charge on the left. Besides infinity, where else can electric field possibly be zero?

4. A test charge is placed at the point P near a positively charged insulating rod.



How would the magnitude and direction of the electric field change if the magnitude of the test charge were decreased and its sign changed with everything else remaining the same?

5. An electric dipole, consisting of two equal and opposite point charges at the ends of an insulating rod, is free to rotate about a pivot point in the center. The rod is then placed in a non-uniform electric field. Does it experience a force and/or a torque?

2.16 Additional Problems

2.16.1 Three Point-Like Charged Objects on Vertices of Equilateral Triangle

Three point-like charged objects are placed at the corners of an equilateral triangle, as shown in Figure 2.16.1. Calculate the electric force experienced by (a) the $+9.00\ \mu\text{C}$ charge, and (b) the $-6.00\ \mu\text{C}$ charge.

2.16.2 Three Point-Like Charged Objects on Vertices of Right Triangle

A right isosceles triangle of side a has charges $+q$, $+2q$, and $-q$ arranged on its vertices, as shown in Figure 2.16.2. What is the electric field at point *P*, midway between the line connecting the $+q$ and $-q$ charges? Give the magnitude and direction of the electric field.

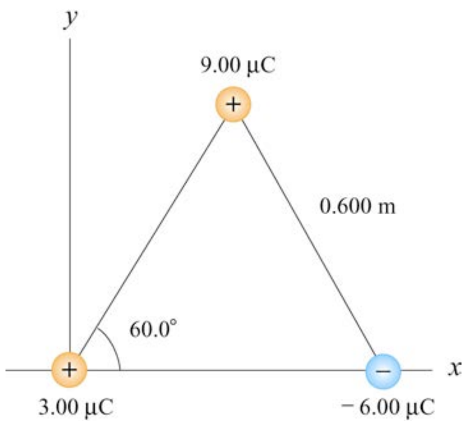


Figure 2.16.1 Three point-like charged objects

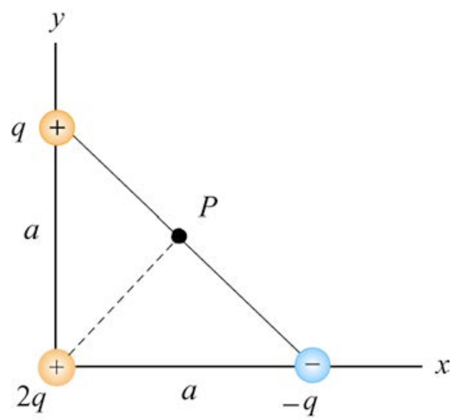


Figure 2.16.2

2.16.3 Four Point-Like Charged Objects

Four point-like charged objects are placed at the corners of a square of side a, as shown in Figure 2.16.3.

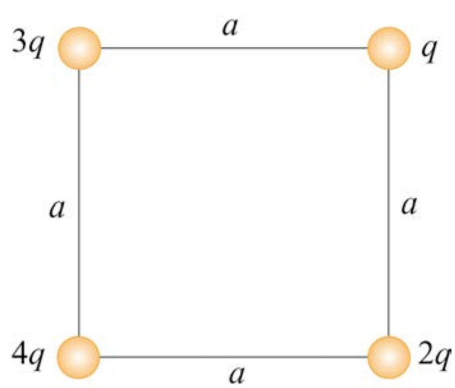


Figure 2.16.3 Four point-like charged objects

(a) What is the electric field at the location of charge q?

(b) What is the electric force on $2q$?

2.16.4 Semicircular Wire

A positively charged wire is bent into a semicircle of radius R, as shown in Figure 2.16.4. The charge on the semicircle is Q. However, the charge per unit length along the semicircle is non-uniform and given by $\lambda(\theta) = \lambda_0 \cos\theta$.

(a) What is the relationship between λ_0, R and Q?

(b) If a point-like charged object with charge q is placed at the origin, what is the force on that object?

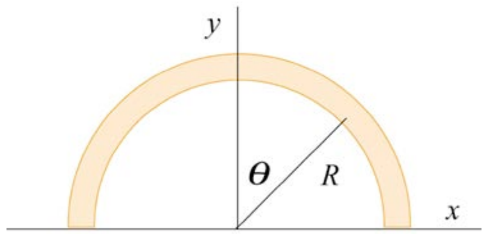


Figure 2.16.4

2.16.5 Electric Dipole

An electric dipole lying in the xy-plane with a uniform electric field applied in the positive x-direction is displaced by a small angle θ from its equilibrium position, as shown in Figure 2.16.5.

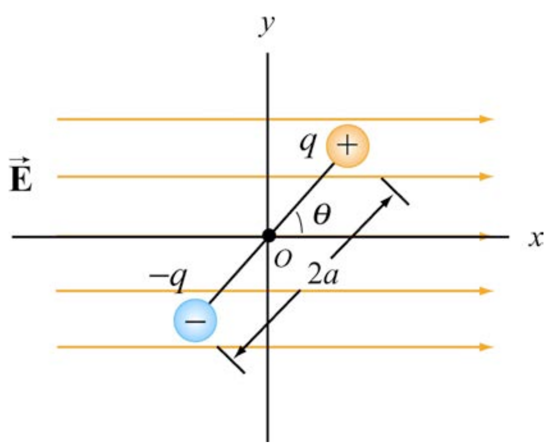


Figure 2.16.5

The charges are separated by a distance $2a$. The moment of inertia of the dipole about the center of mass is I_{cm}. If the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion. What is the period of oscillation?

2.16.6 Charged Cylindrical Shell and Cylinder

(a) A uniformly charged circular *cylindrical shell* of radius R and height h has a total charge Q. What is the electric field at a point P a distance z from the bottom side of the cylinder as shown in Figure 2.16.6? (*Hint:* Treat the cylinder as a set of charged rings.)

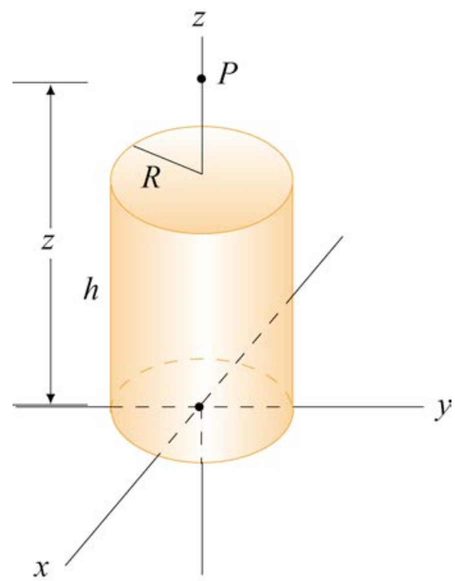


Figure 2.16.6 A uniformly charged cylinder

(b) If the configuration is instead a *solid cylinder* of radius R, height h and has a uniform volume charge density. What is the electric field at P? (*Hint:* Treat the solid cylinder as a set of charged disks.)

2.16.7 Two Conducting Balls

Two tiny conducting balls of identical mass m and identical charge q hang from non-conducting threads of length l. Each ball forms an angle θ with the vertical axis, as shown in Figure 2.16.9. Assume that θ is so small that $\tan\theta \approx \sin\theta$.

(a) Show that, at equilibrium, the separation between the balls is $r = (q^2 l / 2\pi\varepsilon_0 mg)^{1/3}$.

(b) If $l = 1.2\times10^2$ cm, $m = 1.0\times10^1$ g, and $x = 5.0$ cm, what is q?

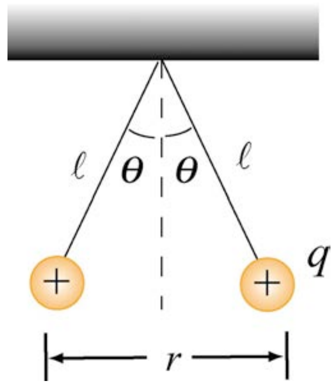


Figure 2.16.9

2.16.8 Torque on an Electric Dipole

An electric dipole consists of two charges $q_1 = +2e$ and $q_2 = -2e$ (where $e = 1.6\times10^{-19}\,\text{C}$), separated by a distance $d = 10^{-9}\,\text{m}$. The electric charges are placed along the y-axis as shown in Figure 2.16.10. Suppose a constant external electric field $\vec{\mathbf{E}}_{\text{ext}} = (3\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}})\,\text{N/C}$ is applied.

(a) What is the magnitude and direction of the dipole moment?

(b) What is the magnitude and direction of the torque on the dipole?

(c) Do the electric fields of the charges q_1 and q_2 contribute to the torque on the dipole? Briefly explain your answer.

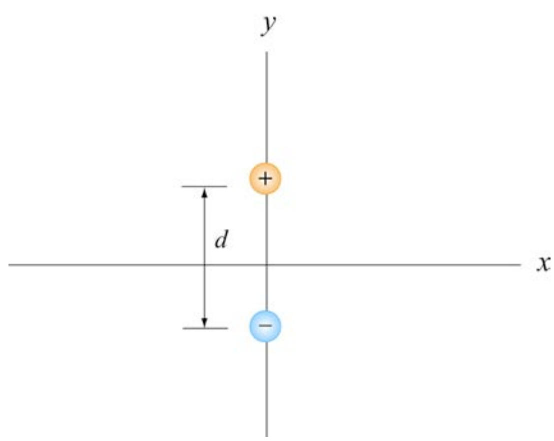


Figure 2.16.10