Chapter 12

Driven *RLC* Circuits

Driven *RLC* Circuits

12.1 AC Sources

In Chapter 10 we learned that changing magnetic flux could induce an emf according to Faraday's law of induction. In particular, if a coil rotates in the presence of a magnetic field, the induced emf varies sinusoidally with time and leads to an *alternating current* (AC), and provides a source of AC power. The symbol for an AC voltage source is

An example of an AC source is

$$V(t) = V_0 \sin(\omega t)\,, \tag{12.1.1}$$

where the maximum value V_0 is called the *amplitude*. The voltage varies between V_0 and $-V_0$ since a sine function varies between +1 and −1. A graph of voltage as a function of time is shown in Figure 12.1.1. The *phase* of the voltage source is $\phi_V = \omega t$, (the phase constant is zero in Eq. (12.1.1)).

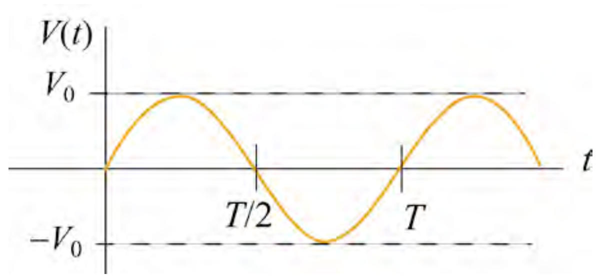


Figure 12.1.1 Sinusoidal voltage source

The sine function is periodic in time. This means that the value of the voltage at time t will be exactly the same at a later time $t' = t + T$ where T is the *period*. The *frequency*, f, defined as $f = 1/T$, has the unit of inverse seconds (s^{-1}), or hertz (Hz). The *angular frequency* is defined to be $\omega = 2\pi f$.

When a voltage source is connected to a *RLC* circuit, energy is provided to compensate the energy dissipation in the resistor, and the oscillation will no longer damp out. The oscillations of charge, current and potential difference are called *driven or forced oscillations.*

After an initial "transient time," an AC current will flow in the circuit as a response to the driving voltage source. The current in the circuit is also sinusoidal,

$$I(t) = I_0 \sin(\omega t - \phi), \qquad (12.1.2)$$

and will oscillate with the same angular frequency ω as the voltage source, has amplitude I_0, phase $\phi_I = \omega t - \phi$, and phase constant ϕ that depends on the driving angular frequency. Note that the phase constant is equal to the *phase difference* between the voltage source and the current

$$\Delta\phi \equiv \phi_V - \phi_I = \omega t - (\omega t - \phi) = \phi. \qquad (12.1.3)$$

12.2 AC Circuits with a Source and One Circuit Element

Before examining the driven *RLC* circuit, let's first consider cases where only one circuit element (a resistor, an inductor or a capacitor) is connected to a sinusoidal voltage source.

12.2.1 Purely Resistive Load

Consider a purely resistive circuit with a resistor connected to an AC generator with AC source voltage given by $V(t) = V_0 \sin(\omega t)$, as shown in Figure 12.2.1. (As we shall see, a purely resistive circuit corresponds to infinite capacitance $C = \infty$ and zero inductance $L = 0$.)

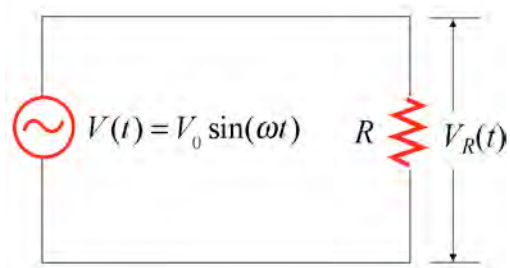


Figure 12.2.1 A purely resistive circuit

We would like to find the current through the resistor,

$$I_R(t) = I_{R0} \sin(\omega t - \phi_R). \qquad (12.2.1)$$

Applying Kirchhoff's loop rule yields

$$V(t) - I_R(t) R = 0, \qquad (12.2.2)$$

where $V_R(t) = I_R(t) R$ is the instantaneous voltage drop across the resistor. The instantaneous current in the resistor is given by

$$I_R(t) = \frac{V(t)}{R} = \frac{V_0 \sin(\omega t)}{R} = I_{R0} \sin(\omega t) . \tag{12.2.3}$$

Comparing Eq. (12.2.3) with Eq. (12.2.1), we find that the amplitude is

$$I_{R0} = \frac{V_{R0}}{R} = \frac{V_{R0}}{X_R} \tag{12.2.4}$$

where $V_{R0} = V_0$, and

$$X_R = R . \tag{12.2.5}$$

The quantity X_R is called the *resistive reactance*, to be consistent with nomenclature that will introduce shortly for capacitive and inductive elements, but it is just the resistance. The key point to recognize is that the amplitude of the current is independent of the driving angular frequency. Because $\phi_R = 0$, $I_R(t)$ and $V_R(t)$ are in phase with each other, i.e. they reach their maximum or minimum values at the same time, the phase constant is zero,

$$\phi_R = 0 . \tag{12.2.6}$$

The time dependence of the current and the voltage across the resistor is depicted in Figure 12.2.2(a).

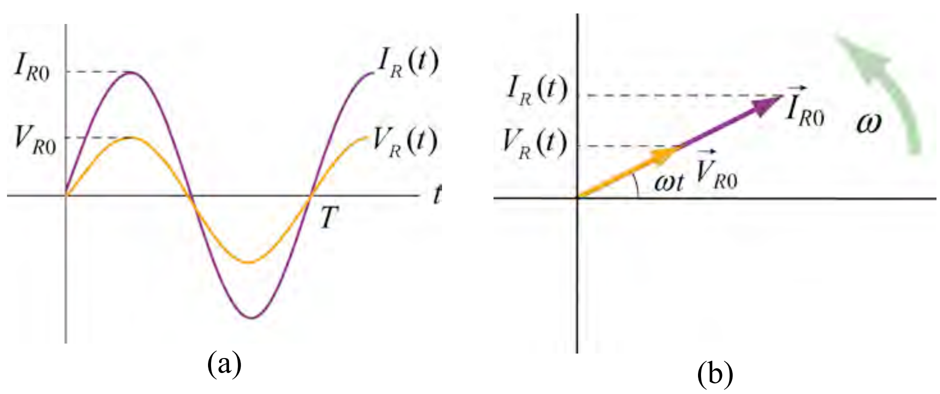


Figure 12.2.2 (a) Time dependence of $I_R(t)$ and $V_R(t)$ across the resistor. (b) Phasor diagram for the resistive circuit.

The behavior of $I_R(t)$ and $V_R(t)$ can also be represented with a *phasor diagram*, as shown in Figure 12.2.2(b). A *phasor* is a rotating vector having the following properties;

(i) length: the length corresponds to the amplitude.

(ii) angular speed: the vector rotates counterclockwise with an angular speed ω.

(iii) projection: the projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time *t*.

We shall denote a phasor with an arrow above it. The phasor $\vec{V}_{R0}$ has a constant magnitude of V_{R0}. Its projection along the vertical direction is $V_{R0}\sin(\omega t)$, which is equal to $V_R(t)$, the voltage drop across the resistor at time t. A similar interpretation applies to $\vec{I}_{R0}$ for the current passing through the resistor. From the phasor diagram, we readily see that both the current and the voltage are in phase with each other.

The average value of current over one period can be obtained as:

$$\left\langle I_R(t)\right\rangle = \frac{1}{T}\int_0^T I_R(t)dt = \frac{1}{T}\int_0^T I_{R0}\sin(\omega t)\,dt = \frac{I_{R0}}{T}\int_0^T \sin\left(\frac{2\pi t}{T}\right)dt = 0\,. \quad (12.2.7)$$

This average vanishes because

$$\left\langle \sin(\omega t)\right\rangle = \frac{1}{T}\int_0^T \sin(\omega t)\,dt = 0\,. \quad (12.2.8)$$

Similarly, one may find the following relations useful when averaging over one period,

$$\begin{aligned}
\left\langle \cos(\omega t)\right\rangle &= \frac{1}{T}\int_0^T \cos(\omega t)\,dt = 0,\\
\left\langle \sin(\omega t)\cos(\omega t)\right\rangle &= \frac{1}{T}\int_0^T \sin(\omega t)\cos(\omega t)\,dt = 0,\\
\left\langle \sin^2(\omega t)\right\rangle &= \frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)dt = \frac{1}{2},\\
\left\langle \cos^2(\omega t)\right\rangle &= \frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = \frac{1}{T}\int_0^T \cos^2\left(\frac{2\pi t}{T}\right)dt = \frac{1}{2}.
\end{aligned} \quad (12.2.9)$$

From the above, we see that the average of the square of the current is non-vanishing:

$$\left\langle I_R^2(t)\right\rangle = \frac{1}{T}\int_0^T I_R^2(t)dt = \frac{1}{T}\int_0^T I_{R0}^2\sin^2\omega t\,dt = I_{R0}^2\frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)dt = \frac{1}{2}I_{R0}^2\,. \quad (12.2.10)$$

It is convenient to define the root-mean-square (rms) current as

$$I_{\text{rms}} = \sqrt{\left\langle I_R^2(t)\right\rangle} = \frac{I_{R0}}{\sqrt{2}} \quad (12.2.11)$$

In a similar manner, the rms voltage can be defined as

$$V_{\text{rms}} = \sqrt{\langle V_R^2(t) \rangle} = \frac{V_{R0}}{\sqrt{2}} . \qquad (12.2.12)$$

The rms voltage supplied to the domestic wall outlets in the United States is $V_{\text{rms}} = 120\ \text{V}$ at a frequency $f = 60\ \text{Hz}$.

The power dissipated in the resistor is

$$P_R(t) = I_R(t) V_R(t) = I_R^2(t) R . \qquad (12.2.13)$$

The average power over one period is then

$$\langle P_R(t) \rangle = \langle I_R^2(t) R \rangle = \frac{1}{2} I_{R0}^2 R = I_{\text{rms}}^2 R = I_{\text{rms}} V_{\text{rms}} = \frac{V_{\text{rms}}^2}{R} . \qquad (12.2.14)$$

12.2.2 Purely Inductive Load

Consider now a purely inductive circuit with an inductor connected to an AC generator with AC source voltage given by $V(t) = V_0 \sin(\omega t)$, as shown in Figure 12.2.3. As we shall see below, a purely inductive circuit corresponds to infinite capacitance $C = \infty$ and zero resistance $R = 0$.

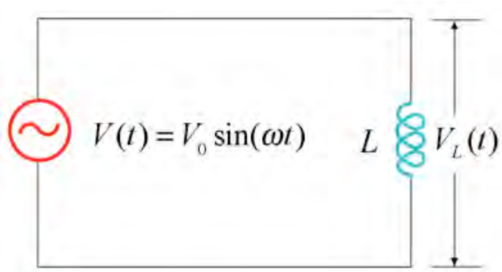


Figure 12.2.3 A purely inductive circuit

We would like to find the current in the circuit,

$$I_L(t) = I_{L0} \sin(\omega t - \phi_L) . \qquad (12.2.15)$$

Applying the modified Kirchhoff's rule for inductors, the circuit equation yields

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0 . \qquad (12.2.16)$$

where we define $V_L(t) = L dI_L / dt$. Rearranging yields

$$\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{L0}}{L}\sin(\omega t), \tag{12.2.17}$$

where $V_{L0} = V_0$. Integrating Eq. (12.2.17), we find the current is

$$\begin{aligned} I_L(t) &= \int dI_L = \frac{V_{L0}}{L}\int \sin(\omega t)\, dt = -\frac{V_{L0}}{\omega L}\cos(\omega t) = \frac{V_{L0}}{\omega L}\sin(\omega t - \pi/2) \\ &= I_{L0}\sin(\omega t - \pi/2). \end{aligned}, \tag{12.2.18}$$

where we have used the trigonometric identity $-\cos(\omega t) = \sin(\omega t - \pi/2)$. Comparing Eq. (12.2.18) with Eq. (12.2.15), we find that the amplitude is

$$I_{L0} = \frac{V_{L0}}{\omega L} = \frac{V_{L0}}{X_L}, \tag{12.2.19}$$

where the *inductive reactance*, X_L, is given by

$$X_L = \omega L. \tag{12.2.20}$$

The inductive reactance has SI units of ohms (Ω), just like resistance. However, unlike resistance, X_L depends linearly on the angular frequency ω. Thus, the inductance reactance to current flow increases with frequency. This is due to the fact that at higher frequencies the current changes more rapidly than it does at lower frequencies. On the other hand, the inductive reactance vanishes as ω approaches zero.

The phase constant, ϕ_L, can also be determined by comparing Eq. (12.2.18) to Eq. (12.2.15), and is given by

$$\phi_L = +\frac{\pi}{2}. \tag{12.2.21}$$

The current and voltage plots and the corresponding phasor diagram are shown in Figure 12.2.4.

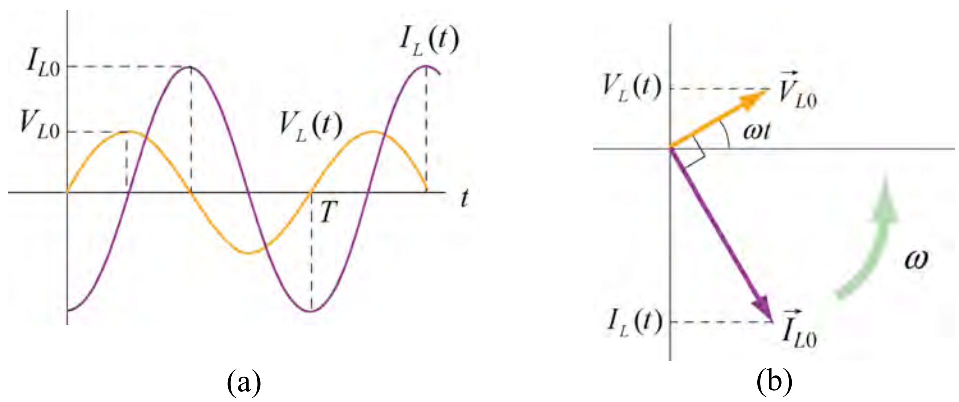


Figure 12.2.4 (a) Time dependence of $I_L(t)$ and $V_L(t)$ across the inductor. (b) Phasor diagram for the inductive circuit.

As can be seen from the figures, the current $I_L(t)$ is out of phase with $V_L(t)$ by $\phi_L = \pi / 2$; it reaches its maximum value one quarter of a cycle later than $V_L(t)$.

The current lags voltage by π / 2 in a purely inductive circuit

The word "lag" means that the plot of $I_L(t)$ is shifted to the right of the plot of $V_L(t)$ in Figure 12.2.4 (a), whereas in the phasor diagram the phasor $\vec{I}_L(t)$ is "behind" the phasor for $\vec{V}_L(t)$ as they rotate counterclockwise in Figure 12.2.4(b).

12.2.3 Purely Capacitive Load

Consider now a purely capacitive circuit with a capacitor connected to an AC generator with AC source voltage given by $V(t) = V_0 \sin(\omega t)$. In the purely capacitive case, both resistance R and inductance L are zero. The circuit diagram is shown in Figure 12.2.5.

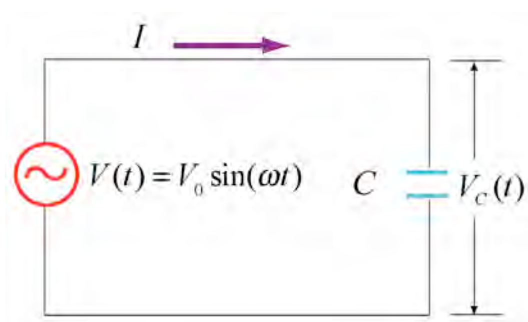


Figure 12.2.5 A purely capacitive circuit

We would like to find the current in the circuit,

$$I_C(t) = I_{C0}\sin(\omega t - \phi_C)\,. \tag{12.2.22}$$

Again, Kirchhoff's loop rule yields

$$V(t) - V_C(t) = V(t) - \frac{Q(t)}{C} = 0\,. \tag{12.2.23}$$

The charge on the capacitor is therefore

$$Q(t) = CV(t) = CV_C(t) = CV_{C0}\sin(\omega t)\,, \tag{12.2.24}$$

where $V_{C0} = V_0$. The current is

$$\begin{aligned} I_C(t) &= +\frac{dQ}{dt} = \omega C V_{C0}\cos(\omega t) = \omega C V_{C0}\sin(\omega t + \pi/2) \\ &= I_{C0}\sin(\omega t + \pi/2), \end{aligned} \tag{12.2.25}$$

where we have used the trigonometric identity $\cos\omega t = \sin(\omega t + \pi/2)$. The maximum value of the current can be determined by comparing Eq. (12.2.25) to Eq. (12.2.22),

$$I_{C0} = \omega C V_{C0} = \frac{V_{C0}}{X_C}\,, \tag{12.2.26}$$

where the *capacitance reactance*, X_C, is

$$X_C = \frac{1}{\omega C}\,. \tag{12.2.27}$$

The capacitive reactance also has SI units of ohms and represents the "effective resistance" for a purely capacitive circuit. Note that X_C is inversely proportional to both C and ω, and diverges as ω approaches zero.

The phase constant can be determined by comparing Eq. (12.2.25) to Eq. (12.2.22), and is

$$\phi_C = -\frac{\pi}{2}\,. \tag{12.2.28}$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 12.2.6 below.

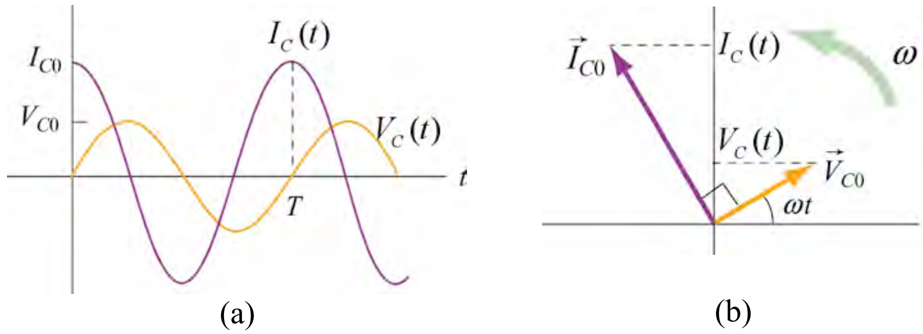


Figure 12.2.6 (a) Time dependence of $I_C(t)$ and $V_C(t)$ across the capacitor. (b) Phasor diagram for the capacitive circuit.

Notice that at $t = 0$, the voltage across the capacitor is zero while the current in the circuit is at a maximum. In fact, $I_C(t)$ reaches its maximum one quarter of a cycle earlier than $V_C(t)$.

The current leads the voltage by $\pi/2$ in a capacitive circuit

The word "lead" means that the plot of $I_C(t)$ is shifted to the left of the plot of $V_C(t)$ in Figure 12.2.6 (a), whereas in the phasor diagram the phasor $\vec{I}_C(t)$ is "ahead" the phasor for $\vec{V}_C(t)$ as they rotate counterclockwise in Figure 12.2.6(b).

12.3 The *RLC* Series Circuit

Consider now the driven series *RLC* circuit with $V(t) = V_0 \sin(\omega t + \phi)$ shown in Figure 12.3.1.

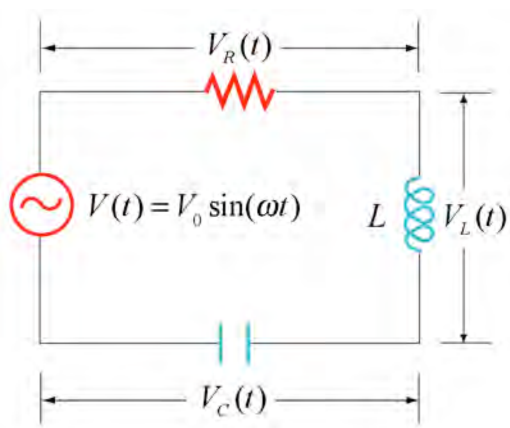


Figure 12.3.1 Driven series *RLC* Circuit

We would like to find the current in the circuit,

$$I(t) = I_0 \sin(\omega t)\,. \tag{12.3.1}$$

Notice that we have added a phase constant ϕ to our previous expressions for $V(t)$ and $I(t)$ when we were analyzing single element driven circuits. Applying Kirchhoff's modified loop rule, we obtain

$$V(t) - V_R(t) - V_L(t) - V_C(t) = 0\,. \tag{12.3.2}$$

We can rewrite Eq. (12.3.2) using $V_R(t) = IR$, $V_L(t) = L dI / dt$, and $V_C(t) = Q / C$ as

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = V_0 \sin(\omega t + \phi)\,. \tag{12.3.3}$$

Differentiate Eq. (12.3.3), using $I = +dQ/dt$, and divide through by L, yields what is called a *second order damped linear driven differential equation,*

$$\boxed{\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I}{LC} = \frac{\omega V_0}{L}\cos(\omega t + \phi)}. \tag{12.3.4}$$

We shall find the amplitude, I_0, of the current, and phase constant ϕ which is the phase shift between the voltage source and the current by examining the phasors associates with the three circuit elements R, L and C.

The instantaneous voltages across each of the three circuit elements R, L, and C has a different amplitude and phase compared to the current, as can be seen from the phasor diagrams shown in Figure 12.3.2.

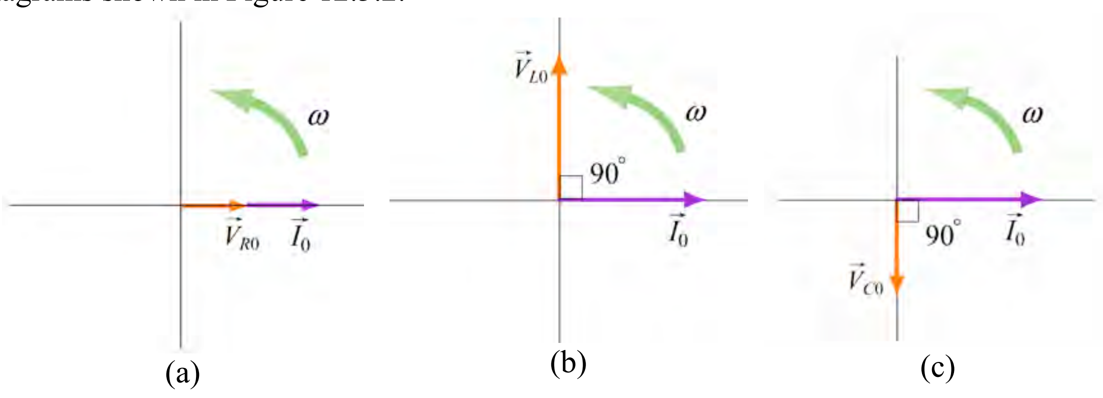


Figure 12.3.2 Phasor diagrams for the relationships between current and voltage in (a) the resistor, (b) the inductor, and (c) the capacitor, of a series RLC circuit.

Using the phasor representation, Eq. (12.3.2) can be written as

$$\vec{V}_0 = \vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0} \tag{12.3.5}$$

as shown in Figure 12.3.3(a). Again we see that current phasor $\vec{I}_0$ leads the capacitive voltage phasor $\vec{V}_{C0}$ by $\pi/2$ but lags the inductive voltage phasor $\vec{V}_{L0}$ by $\pi/2$. The three voltage phasors rotate counterclockwise as time increases, with their relative positions fixed.

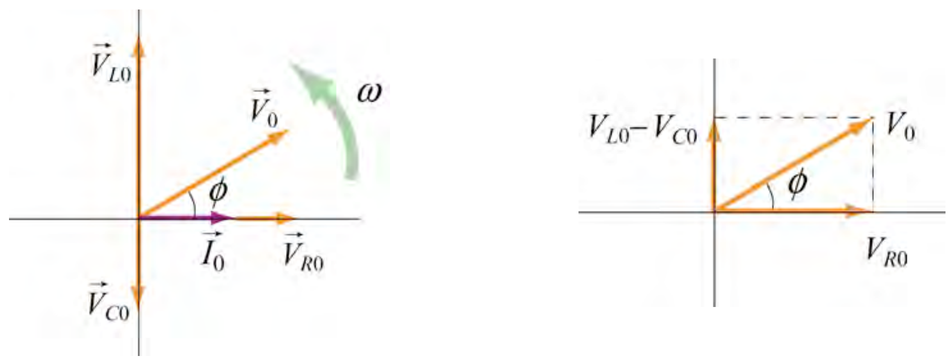


Figure 12.3.3 (a) Phasor diagram for the series *RLC* circuit. (b) voltage relationship

The relationship between different voltage amplitudes is depicted in Figure 12.3.3(b). From Figure 12.3.3, we see that the amplitude satisfies

$$\begin{aligned} V_0 &= |\vec{V}_0| = |\vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0}| = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} \\ &= \sqrt{(I_0 X_R)^2 + (I_0 X_L - I_0 X_C)^2} \\ &= I_0 \sqrt{X_R{}^2 + (X_L - X_C)^2}. \end{aligned} \tag{12.3.6}$$

Therefore the amplitude of the current is

$$I_0 = \frac{V_0}{\sqrt{X_R{}^2 + (X_L - X_C)^2}}. \tag{12.3.7}$$

Using Eqs. (12.2.5), (12.2.20), and (12.2.27) for the reactances, Eq. (12.3.7) becomes

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - \dfrac{1}{\omega C})^2}}, \quad \text{series } RLC \text{ circuit}. \tag{12.3.8}$$

From Figure 12.3.3(b), we can determine that the phase constant satisfies

$$\tan\phi = \left(\frac{X_L - X_C}{X_R}\right) = \frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right). \tag{12.3.9}$$

Therefore the phase constant is

$$\phi = \tan^{-1}\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right), \quad \text{series } RLC \text{ circuit}. \tag{12.3.10}$$

It is crucial to note that the maximum amplitude of the AC voltage source V_0 is not equal to the sum of the maximum voltage amplitudes across the three circuit elements:

$$V_0 \neq V_{R0} + V_{L0} + V_{C0} \tag{12.3.11}$$

This is due to the fact that the voltages are not in phase with one another, and they reach their maxima at different times.

12.3.1 Impedance

We have already seen that the inductive reactance $X_L = \omega L$, and the capacitive reactance $X_C = 1/\omega C$ play the role of an effective resistance in the purely inductive and capacitive circuits, respectively. In the series RLC circuit, the effective resistance is the *impedance*, defined as

$$Z = \sqrt{X_R^{\,2} + (X_L - X_C)^2} \tag{12.3.12}$$

The relationship between Z, X_R, X_L, and X_C can be represented by the diagram shown in Figure 12.3.4: $X_L - X_C$

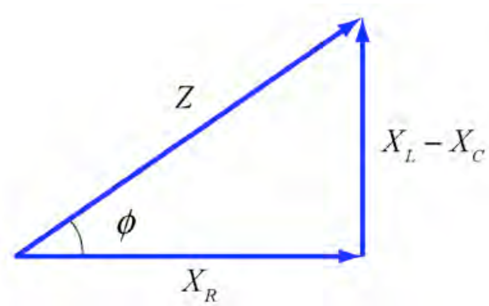


Figure 12.3.4 Vector representation of the relationship between Z, X_R, X_L, and X_C.

The impedance also has SI units of ohms. In terms of Z, the current (Eqs. (12.3.1) and (12.3.7)) may be rewritten as

$$I(t)=\frac{V_0}{Z}\sin(\omega t) \tag{12.3.13}$$

Notice that the impedance Z also depends on the angular frequency ω, as do X_L and X_C.

Using Eq. (12.3.9) for the phase constant ϕ and Eq. (12.3.12) for Z, we may readily recover the limits for simple circuit (with only one element). A summary is provided in Table 12.1 below:

Simple Circuit	R	L	C	$X_L=\omega L$	$X_C=\frac{1}{\omega C}$	$\phi=\tan^{-1}\left(\frac{X_L-X_C}{X_R}\right)$	$Z=\sqrt{X_R{}^2+(X_L-X_C)^2}$
purely resistive	R	0	∞	0	0	0	X_R
purely inductive	0	L	∞	X_L	0	$\pi/2$	X_L
purely capacitive	0	0	C	0	X_C	$-\pi/2$	X_C

Table 12.1 Simple-circuit limits of the series *RLC* circuit

12.3.2 Resonance

In a driven *RLC* series circuit, the amplitude of the current (Eq. (12.3.8)) has a maximum value, a *resonance*, which occurs at the *resonant angular frequency* ω_0. Because the amplitude I_0 of the current is inversely proportionate to Z (Eq. (12.3.13), the maximum of I_0 occurs when Z is minimum. This occurs at an angular frequency ω_0 such that $X_L=X_C$,

$$\omega_0 L=\frac{1}{\omega_0 C}\,. \tag{12.3.14}$$

Therefore, the resonant angular frequency is

$$\omega_0=\frac{1}{\sqrt{LC}}\,. \tag{12.3.15}$$

At resonance, the impedance becomes $Z=R$, and the amplitude of the current is

$$I_0=\frac{V_0}{R}, \tag{12.3.16}$$

and the phase constant is zero, (Eq. (12.3.10)),

$$\phi = 0 \,. \tag{12.3.17}$$

A qualitative plot of the amplitude of the current as a function of driving angular frequency for two driven *RLC* circuits, with different values of resistance, $R_2 > R_1$ is illustrated in Figure 12.3.5. The amplitude is larger for smaller a smaller value of resistance.

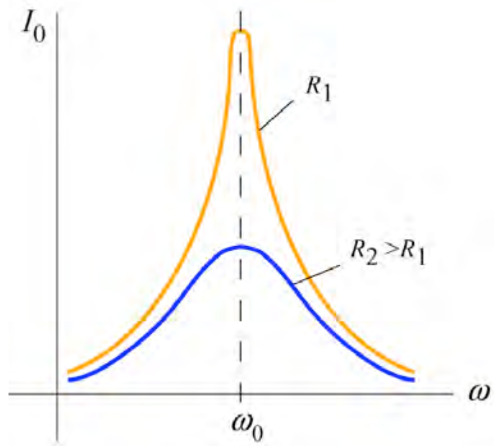


Figure 12.3.5 The amplitude of the current as a function of ω in the driven *RLC* circuit, for two different values of the resistance.

12.4 Power in an AC circuit

In the series *RLC* circuit, the instantaneous power delivered by the AC generator is given by

$$\begin{aligned} P(t) = I(t)V(t) &= \frac{V_0}{Z}\sin(\omega t)\cdot V_0\sin(\omega t+\phi) = \frac{V_0^2}{Z}\sin(\omega t)\sin(\omega t+\phi) \\ &= \frac{V_0^2}{Z}(\sin^2(\omega t)\cos\phi + \sin(\omega t)\cos(\omega t)\sin\phi) \end{aligned} \tag{12.4.1}$$

where we have used the trigonometric identity

$$\sin(\omega t+\phi) = \sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi \,. \tag{12.4.2}$$

The time average of the power is

$$\left\langle P(\omega)\right\rangle = \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin^2(\omega t)\cos\phi\, dt + \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin(\omega t)\cos(\omega t)\sin\phi\, dt = \frac{1}{2}\frac{V_0^2}{Z}\cos\phi \,. \tag{12.4.3}$$

where we have used the integral results in Eq. (12.2.9). In terms of the rms quantities, $V_{\text{rms}} = V_0/\sqrt{2}$ and $I_{\text{rms}} = V_{\text{rms}}/Z$, the time-averaged power can be written as

$$\langle P(\omega) \rangle = \frac{1}{2}\frac{V_0^2}{Z}\cos\phi = \frac{V_{\text{rms}}^2}{Z}\cos\phi = I_{\text{rms}}V_{\text{rms}}\cos\phi \tag{12.4.4}$$

The quantity $\cos\phi$ is called the *power factor*. From Figure 12.3.4, one can readily show that

$$\cos\phi = \frac{R}{Z}. \tag{12.4.5}$$

Thus, we may rewrite $\langle P(\omega) \rangle$ as

$$\langle P(\omega) \rangle = I_{\text{rms}}^2(\omega)R, \tag{12.4.6}$$

where

$$I_{\text{rms}}(\omega) = \frac{1}{\sqrt{2}}\frac{V_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}, \tag{12.4.7}$$

In Figure 12.4.1, we plot the time-averaged power as a function of the driving angular frequency ω for two driven RLC circuits, with different values of resistance, $R_2 > R_1$.

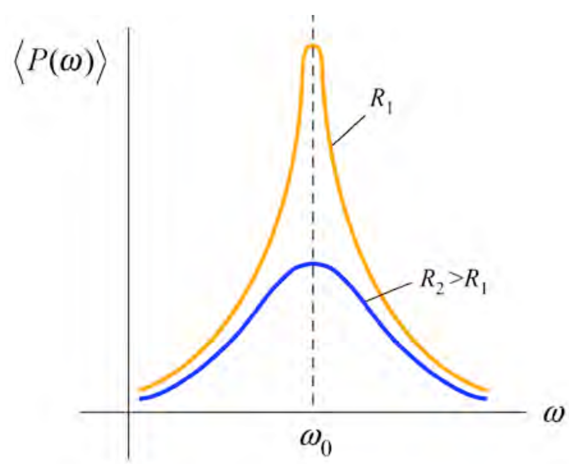


Figure 12.4.1 Average power as a function of frequency in a driven series RLC circuit.

We see that $\langle P(\omega) \rangle$ attains the maximum value when $\cos\phi = 1$, or $Z = R$, which is the resonance condition. At resonance, we have

$$\langle P(\omega_0) \rangle = I_{\text{rms}}V_{\text{rms}} = \frac{V_{\text{rms}}^2}{R}. \tag{12.4.8}$$

12.4.1 Width of the Peak

The peak has a line width. One way to characterize the width is to define $\Delta\omega = \omega_+ - \omega_-$, where $\omega_\pm$ are the values of the driving angular frequency such that the power is equal to half its maximum power at resonance. This is called *full width at half maximum*, as illustrated in Figure 12.4.2. The width $\Delta\omega$ increases with resistance R.

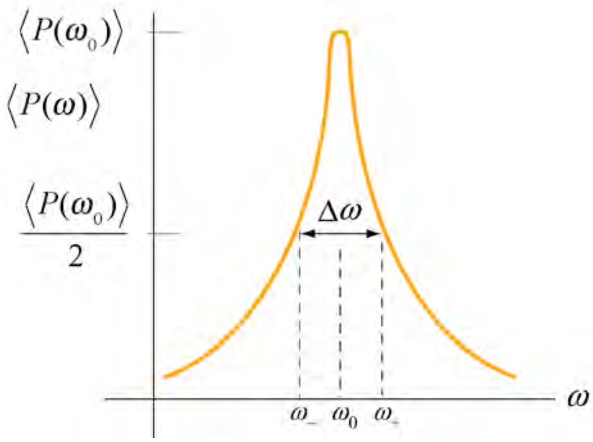


Figure 12.4.2 Width of the peak

To find $\Delta\omega$, it is instructive to first rewrite the average power $\langle P(\omega)\rangle$ as

$$\langle P(\omega)\rangle = \frac{1}{2}\frac{V_0^{\,2}R}{R^2 + (\omega L - 1/\omega C)^2} = \frac{1}{2}\frac{V_0^{\,2}R\omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2}, \tag{12.4.9}$$

with $\langle P(\omega_0)\rangle = V_0^2 / 2R$. The condition for finding $\omega_\pm$ is

$$\frac{1}{2}\langle P((\omega_0))\rangle = \langle P(\omega_\pm)\rangle \;\Rightarrow\; \frac{V_0^{\,2}}{4R} = \frac{1}{2}\frac{V_0^{\,2}R\omega_\pm^{\,2}}{\omega_\pm^{\,2}R^2 + L^2(\omega_\pm^{\,2} - \omega_0^2)^2}. \tag{12.4.10}$$

Eq. (12.4.10) reduces to

$$(\omega^2 - \omega_0^2)^2 = \left(\frac{R\omega}{L}\right)^2. \tag{12.4.11}$$

Taking square roots yields two solutions, which we analyze separately.

Case 1: Taking the positive root leads to

$$\omega_+^2 - \omega_0^2 = +\frac{R\omega_+}{L} \tag{12.4.12}$$

We can solve this quadratic equation, and the solution with positive root is

$$\omega_+ = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_0{}^2} \,. \tag{12.4.13}$$

Case 2: Taking the negative root of Eq. (12.4.12) yields

$$\omega_-^2 - \omega_0^2 = -\frac{R\omega_-}{L} \,. \tag{12.4.14}$$

The solution to this quadratic equation with positive root is

$$\omega_- = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \omega_0{}^2} \,. \tag{12.4.15}$$

The width at half maximum is then the difference

$$\Delta\omega = \omega_+ - \omega_- = \frac{R}{L} \,. \tag{12.4.16}$$

Once the width $\Delta\omega$ is known, the *quality factor* Q_{qual} is defined as

$$\boxed{Q_{\text{qual}} = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}} \,. \tag{12.4.17}$$

Comparing the above equation with Eq. (11.10.17), we see that both expressions agree with each other in the limit where the resistance is small, and $\omega' = \sqrt{\omega_0^2 - (R/2L)^2} \approx \omega_0$.

12.5 Transformer

A transformer is a device used to increase or decrease the AC voltage in a circuit. A typical device consists of two coils of wire, a primary and a secondary, wound around an iron core, as illustrated in Figure 12.5.1. The primary coil, with N_1 turns, is connected to alternating voltage source $V_1(t)$. The secondary coil has N_2 turns and is connected to a load with resistance R_2. The way transformers operate is based on the principle that an alternating current in the primary coil will induce an alternating emf on the secondary coil due to their mutual inductance.

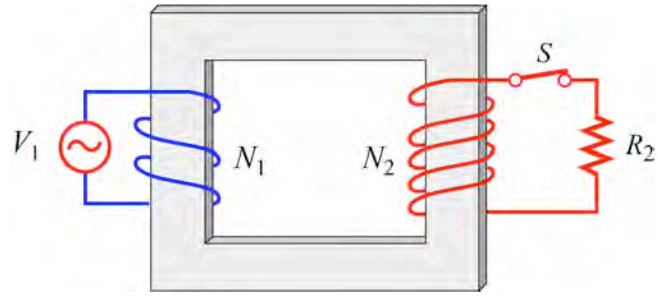


Figure 12.5.1 A transformer

In the primary circuit, neglecting the small resistance in the coil, Faraday's law of induction implies

$$V_1 = -N_1 \frac{d\Phi_B}{dt}, \tag{12.5.1}$$

where Φ_B is the magnetic flux through one turn of the primary coil. The iron core, which extends from the primary to the secondary coils, serves to increase the magnetic field produced by the current in the primary coil and ensures that nearly all the magnetic flux through the primary coil also passes through each turn of the secondary coil. Thus, the voltage (or induced emf) across the secondary coil is

$$V_2 = -N_2 \frac{d\Phi_B}{dt}. \tag{12.5.2}$$

In the case of an ideal transformer, power loss due to Joule heating can be ignored, so that the power supplied by the primary coil is completely transferred to the secondary coil,

$$I_1 V_1 = I_2 V_2. \tag{12.5.3}$$

In addition, no magnetic flux leaks out from the iron core, and the flux Φ_B through each turn is the same in both the primary and the secondary coils. Combining the two expressions, we are lead to the transformer equation,

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}}. \tag{12.5.4}$$

By combining the two equations above, the transformation of currents in the two coils may be obtained as

$$I_1 = \frac{V_2}{V_1} I_2 = \frac{N_2}{N_1} I_2. \tag{12.5.5}$$

Thus, we see that the ratio of the output voltage to the input voltage is determined by the *turn ratio* N_2 / N_1. If $N_2 > N_1$, then $V_2 > V_1$, which means that the output voltage in the second coil is greater than the input voltage in the primary coil. A transformer with $N_2 > N_1$ is called a *step-up* transformer. On the other hand, if $N_2 < N_1$, then $V_2 < V_1$, and the output voltage is smaller than the input. A transformer with $N_2 < N_1$ is called a *step-down* transformer.

12.6 Parallel *RLC* Circuit

Consider the parallel RLC circuit illustrated in Figure 12.6.1. The AC voltage source is $V(t) = V_0 \sin(\omega t)$.

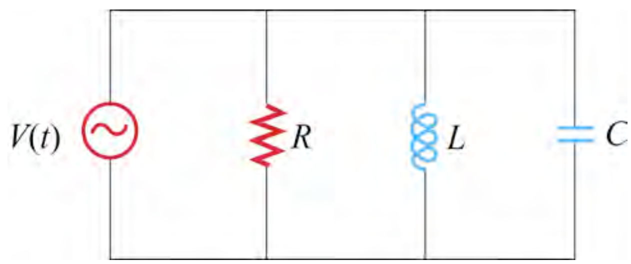


Figure 12.6.1 Parallel RLC circuit.

Unlike the series RLC circuit, the instantaneous voltages across all three circuit elements R, L, and C are the same, and each voltage is in phase with the current through the resistor. However, the currents through each element will be different.

In analyzing this circuit, we make use of the results discussed in Sections 12.2 – 12.4. The current in the resistor is

$$I_R(t) = \frac{V(t)}{R} = \frac{V_0}{R}\sin(\omega t) = I_{R0}\sin(\omega t). \tag{12.6.1}$$

where $I_{R0} = V_0 / R$. The voltage across the inductor is

$$V_L(t) = V(t) = V_0 \sin(\omega t) = L\frac{dI_L}{dt}. \tag{12.6.2}$$

Integrating Eq. (12.6.2) yields

$$I_L(t) = \int_0^t \frac{V_0}{L}\sin(\omega t')dt' = -\frac{V_0}{\omega L}\cos(\omega t) = \frac{V_0}{X_L}\sin(\omega t - \pi/2) = I_{L0}\sin(\omega t - \pi/2), \tag{12.6.3}$$

where $I_{L0} = V_0 / X_L$ and $X_L = \omega L$ is the inductive reactance.

Similarly, the voltage across the capacitor is $V_C(t) = V_0 \sin(\omega t) = Q(t)/C$, which implies

$$I_C(t) = \frac{dQ}{dt} = \omega C V_0 \cos(\omega t) = \frac{V_0}{X_C}\sin(\omega t + \pi/2) = I_{C0}\sin(\omega t + \pi/2), \quad (12.6.4)$$

where $I_{C0} = V_0 / X_C$ and $X_C = 1/\omega C$ is the capacitive reactance.

Using Kirchhoff's junction rule, the total current in the circuit is simply the sum of all three currents.

$$\begin{aligned} I(t) &= I_R(t) + I_L(t) + I_C(t) \\ &= I_{R0}\sin(\omega t) + I_{L0}\sin(\omega t - \pi/2) + I_{C0}\sin(\omega t + \pi/2). \end{aligned} \quad (12.6.5)$$

The currents can be represented with the phasor diagram shown in Figure 12.6.2.

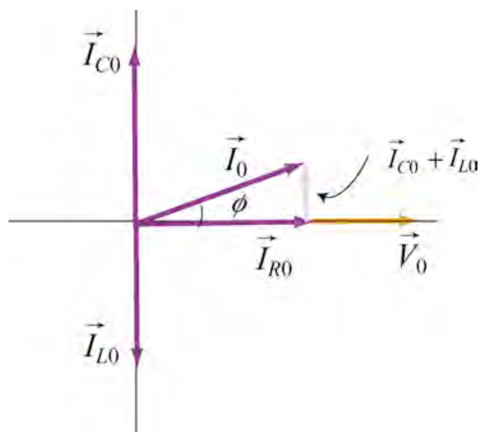


Figure 12.6.2 Phasor diagram for the parallel *RLC* circuit

From the phasor diagram, the phasors satisfy the vector addition condition,

$$\vec{I}_0 = \vec{I}_{R0} + \vec{I}_{L0} + \vec{I}_{C0}. \quad (12.6.6)$$

The amplitude of the current, I_0, can be obtained as

$$\begin{aligned} I_0 &= |\vec{I}_0| = |\vec{I}_{R0} + \vec{I}_{L0} + \vec{I}_{C0}| = \sqrt{I_{R0}^2 + (I_{C0} - I_{L0})^2} \\ &= V_0\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = V_0\sqrt{\frac{1}{X_R{}^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2} \end{aligned} \quad . \quad (12.6.7)$$

Therefore the amplitude of the current is

$$I_0 = V_0\sqrt{\frac{1}{X_R^{\,2}} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2} \quad , \quad \text{parallel } RLC \text{ circuit}. \tag{12.6.8}$$

From the phasor diagram shown in Figure 12.6.2, we see that the tangent of the phase constant can be obtained as

$$\tan\phi = \left(\frac{I_{C0} - I_{L0}}{I_{R0}}\right) = \frac{\dfrac{V_0}{X_C} - \dfrac{V_0}{X_L}}{\dfrac{V_0}{R}} = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right) = R\left(\omega C - \frac{1}{\omega L}\right). \tag{12.6.9}$$

Therefore the phase constant is

$$\phi = \tan^{-1}\left(R\omega C - \frac{R}{\omega L}\right), \quad \text{parallel } RLC \text{ circuit}. \tag{12.6.10}$$

Because $I_R(t)$, $I_L(t)$ and $I_C(t)$ are not in phase with one another, I_0 is not equal to the sum of the amplitudes of the three currents,

$$I_0 \neq I_{R0} + I_{L0} + I_{C0}. \tag{12.6.11}$$

With $I_0 = V_0 / Z$, the (inverse) impedance of the circuit is given by

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = \sqrt{\frac{1}{X_R^{\,2}} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2} \tag{12.6.12}$$

The relationship between Z, X_R, X_L and X_C is shown in Figure 12.6.3.

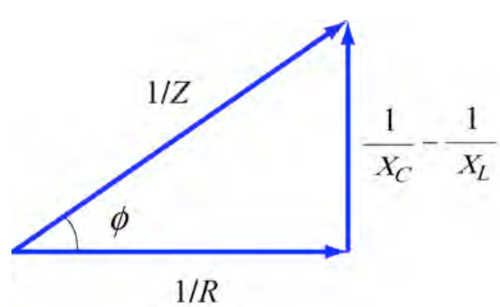


Figure 12.6.3 Relationship between Z, X_R, X_L and X_C in a parallel RLC circuit.

The resonance condition for the parallel RLC circuit is given by $\phi = 0$, which implies

$$\frac{1}{X_C} = \frac{1}{X_L}. \tag{12.6.13}$$

We can solve Eq. (12.6.13) for the resonant angular frequency and find that

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{12.6.14}$$

as in the series RLC circuit. From Eq. (12.6.12), we readily see that $1/Z$ is minimum (or Z is maximum) at resonance. The current in the inductor exactly cancels out the current in the capacitor, so that the current in the circuit reaches a minimum, and is equal to the current in the resistor,

$$I_0 = \frac{V_0}{R} \tag{12.6.15}$$

As in the series RLC circuit, power is dissipated only through the resistor. The time-averaged power is

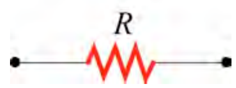
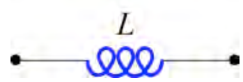
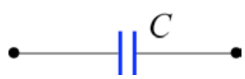
$$\langle P(t) \rangle = \langle I_R(t) V(t) \rangle = \langle I_R^2(t) R \rangle = \frac{V_0^2}{R} \langle \sin^2(\omega t) \rangle = \frac{V_0^2}{2R} = \frac{V_0^2}{2Z} \frac{Z}{R}. \tag{12.6.16}$$

Thus, the power factor in this case is

$$\text{power factor} = \frac{\langle P(t) \rangle}{V_0^2 / 2Z} = \frac{Z}{R} = \frac{1}{\sqrt{1 + \left(R\omega C - \dfrac{R}{\omega L} \right)^2}} = \cos\phi. \tag{12.6.17}$$

12.7 Summary

- In an AC circuit with a sinusoidal voltage source $V(t) = V_0 \sin(\omega t)$, the current is given by $I(t) = I_0 \sin(\omega t - \phi)$, where I_0 is the amplitude and ϕ is the phase constant (phase difference between the voltage source and the current). For simple circuit with only one element (a resistor, a capacitor or an inductor) connected to the voltage source, the results are as follows:

Circuit Elements	Resistance /Reactance	Current Amplitude	Phase constant ϕ
	R	$I_{R0} = \dfrac{V_0}{R}$	0
	$X_L = \omega L$	$I_{L0} = \dfrac{V_0}{X_L}$	$\pi/2$ current lags voltage by $90°$
	$X_C = \dfrac{1}{\omega C}$	$I_{C0} = \dfrac{V_0}{X_C}$	$-\pi/2$ current leads voltage by $90°$

where X_L is the **inductive reactance** and X_C is the **capacitive reactance**.

- For circuits which have more than one circuit element connected in series, the results are

Circuit Elements	Impedance Z	Current Amplitude	Phase constant ϕ
	$\sqrt{R^2 + X_L^2}$	$I_0 = \dfrac{V_0}{\sqrt{R^2 + X_L^2}}$	$0 < \phi < \dfrac{\pi}{2}$
	$\sqrt{R^2 + X_C^2}$	$I_0 = \dfrac{V_0}{\sqrt{R^2 + X_C^2}}$	$-\dfrac{\pi}{2} < \phi < 0$
	$\sqrt{R^2 + (X_L - X_C)^2}$	$I_0 = \dfrac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$	$\phi > 0$ if $X_L > X_C$ $\phi < 0$ if $X_L < X_C$

where Z is the **impedance** Z of the circuit. For a series RLC circuit, we have

$$Z = \sqrt{R^2 + (X_L - X_C)^2} .$$

The phase constant (phase difference between the voltage and the current) in an AC circuit is

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right).$$

- In the parallel RLC circuit, the impedance is given by

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2},$$

  and the phase constant is

$$\phi = \tan^{-1}\left[R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)\right] = \tan^{-1}\left[R\left(\omega C - \frac{1}{\omega L}\right)\right].$$

- The **rms** (root mean square) voltage and current in an AC circuit are given by

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}, \qquad I_{\text{rms}} = \frac{I_0}{\sqrt{2}}.$$

- The average power of an AC circuit is

$$\langle P(t)\rangle = I_{\text{rms}} V_{\text{rms}} \cos\phi\,,$$

  where $\cos\phi$ is known as the **power factor**.

- The **resonant angular frequency** ω_0 is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

  At **resonance**, the current in the series RLC circuit reaches the maximum, but the current in the parallel RLC circuit is at a minimum.

- The **transformer equation** is

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}},$$

  where V_1 is the voltage source in the primary coil with N_1 turns, and V_2 is the output voltage in the secondary coil with N_2 turns. A transformer with $N_2 > N_1$ is called a *step-up* transformer, and a transformer with $N_2 < N_1$ is called a *step-down* transformer.

12.8 Problem-Solving Tips

In this chapter, we have seen how phasors provide a powerful tool for analyzing the AC circuits. Below are some important tips:

1. Keep in mind the phase relationships for simple circuits

   (1) For a resistor, the voltage and the phase are always in phase.
   (2) For an inductor, the current lags the voltage by $90°$.
   (3) For a capacitor, the current leads the voltage by $90°$.

2. When circuit elements are connected in *series*, the instantaneous current is the same for all elements, and the instantaneous voltages across the elements are out of phase. On the other hand, when circuit elements are connected in *parallel*, the instantaneous voltage is the same for all elements, and the instantaneous currents across the elements are out of phase.

3. For series connection, draw a phasor diagram for the voltages. The amplitudes of the voltage drop across all the circuit elements involved should be represented with phasors. In Figure 12.8.1 the phasor diagram for a series *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

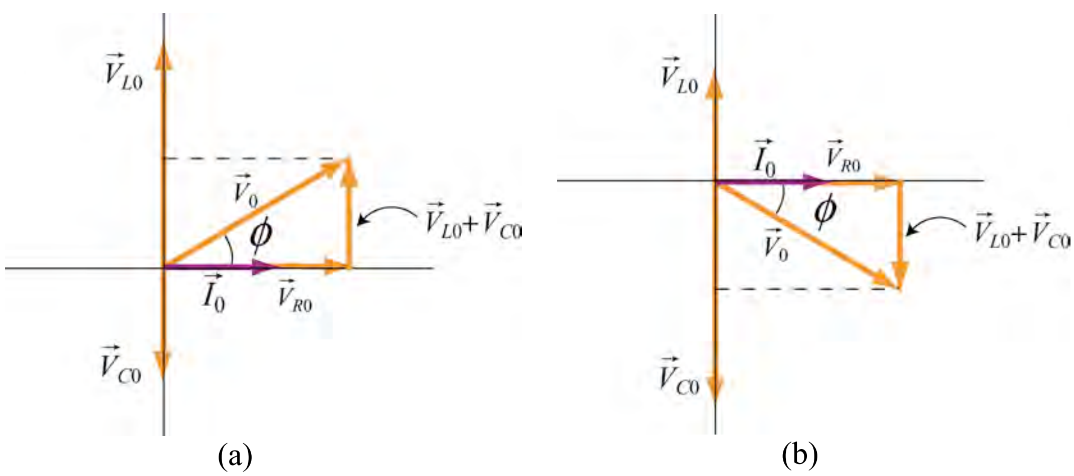


Figure 12.8.1 Phasor diagram for the series *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.1(a), we see that $V_{L0} > V_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase constant ϕ. For the capacitive case shown in Figure 12.8.1(b), $V_{C0} > V_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase constant ϕ.

4. When $V_{L0} = V_{C0}$, or $\phi = 0$, the circuit is at resonance. The corresponding resonant angular frequency is $\omega_0 = 1/\sqrt{LC}$, and the power delivered to the resistor is a maximum.

5. For parallel connection, draw a phasor diagram for the currents. The amplitudes of the currents across all the circuit elements involved should be represented with phasors. In Figure 12.8.2 the phasor diagram for a parallel *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

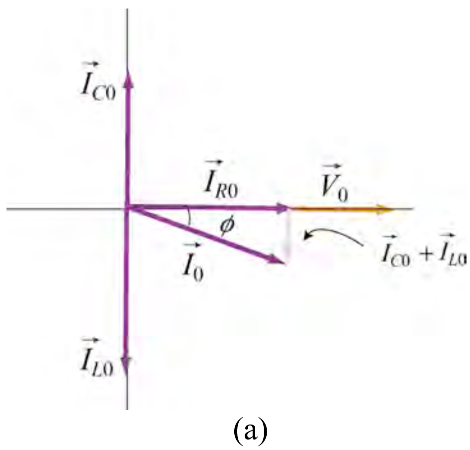


Figure 12.8.2 Phasor diagram for the parallel *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.2(a), we see that $I_{L0} > I_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase constant ϕ. For the capacitive case shown in Figure 12.8.2(b), $I_{C0} > I_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase constant ϕ.

12.9 Solved Problems

12.9.1 *RLC* Series Circuit

A series *RLC* circuit with $L = 160\text{ mH}$, $C = 100\,\mu\text{F}$, and $R = 40.0\,\Omega$ is connected to a sinusoidal voltage $V(t) = (40.0\text{ V})\sin(\omega t)$, with $\omega = 200\text{ rad/s}$.

(a) What is the impedance of the circuit?

(b) Let the current at any instant in the circuit be $I(t) = I_0 \sin(\omega t - \phi)$. Find I_0.

(c) What is the phase constant ϕ?

Solution:

(a) The impedance of a series *RLC* circuit is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}\ , \qquad (12.9.1)$$

where $X_L = \omega L$ and $X_C = 1/\omega C$ are the inductive and capacitive reactances, respectively. The voltage source is $V(t) = V_0 \sin(\omega t)$, where V_0 is the maximum output voltage and ω is the angular frequency, with $V_0 = 40\ \text{V}$ and $\omega = 200\ \text{rad/s}$. Thus, the impedance Z is

$$Z = \sqrt{(40.0\ \Omega)^2 + \left((200\ \text{rad/s})(0.160\ \text{H}) - \frac{1}{(200\ \text{rad/s})(100\times 10^{-6}\ \text{F})}\right)^2} \qquad (12.9.2)$$
$$= 43.9\Omega.$$

(b) With $V_0 = 40.0\,\text{V}$, the amplitude of the current is given by

$$I_0 = \frac{V_0}{Z} = \frac{40.0\,\text{V}}{43.9\,\Omega} = 0.911\,\text{A}\ . \qquad (12.9.3)$$

(c) The phase constant (the phase difference between the voltage source and the current) is given by

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{\omega L - \dfrac{1}{\omega C}}{R}\right)$$
$$= \tan^{-1}\left(\frac{(200\ \text{rad/s})(0.160\ \text{H}) - \dfrac{1}{(200\ \text{rad/s})(100\times 10^{-6}\ \text{F})}}{40.0\ \Omega}\right) = -24.2°. \qquad (12.9.4)$$

12.9.2 *RLC* Series Circuit

Suppose an AC generator with $V(t) = (150\,\text{V})\sin(100t)$ is connected to a series *RLC* circuit with $R = 40.0\ \Omega$, $L = 80.0\ \text{mH}$, and $C = 50.0\ \mu\text{F}$, as shown in Figure 12.9.1.

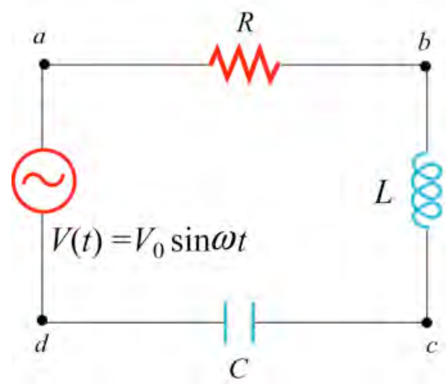


Figure 12.9.1 *RLC* series circuit

(a) Calculate V_{R0}, V_{L0} and V_{C0}, the maximum values of the voltage drop across each circuit element.

(b) Calculate the maximum potential difference across the inductor and the capacitor between points b and d shown in Figure 12.9.1.

Solutions:

(a) The inductive reactance, capacitive reactance and the impedance of the circuit are given by

$$X_C = \frac{1}{\omega C} = \frac{1}{(100\text{ rad/s})(50.0\times 10^{-6}\text{ F})} = 200\ \Omega, \qquad (12.9.5)$$

$$X_L = \omega L = (100\text{ rad/s})(80.0\times 10^{-3}\text{ H}) = 8.00\ \Omega. \qquad (12.9.6)$$

The impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(40.0\ \Omega)^2 + (8.00\ \Omega - 200\ \Omega)^2} = 196\ \Omega. \qquad (12.9.7)$$

The corresponding maximum current amplitude is

$$I_0 = \frac{V_0}{Z} = \frac{150\text{ V}}{196\ \Omega} = 0.765\text{ A}. \qquad (12.9.8)$$

The maximum voltage across the resistance is the product of maximum current and the resistance,

$$V_{R0} = I_0 R = (0.765\text{ A})(40.0\ \Omega) = 30.6\text{ V}. \qquad (12.9.9)$$

Similarly, the maximum voltage across the inductor is

$$V_{L0} = I_0 X_L = (0.765 \text{ A})(8.00 \ \Omega) = 6.12 \text{ V} . \qquad (12.9.10)$$

The maximum voltage across the capacitor is

$$V_{C0} = I_0 X_C = (0.765 \text{ A})(200 \ \Omega) = 153 \text{ V} . \qquad (12.9.11)$$

Note that the maximum input voltage V_0 is related to V_{R0}, V_{L0} and V_{C0} by

$$V_0 = \sqrt{V_{R0}{}^2 + (V_{L0} - V_{C0})^2} . \qquad (12.9.12)$$

(b) From b to d, the maximum voltage would be the difference between V_{L0} and V_{C0},

$$|V_{bd}| = |\vec{V}_{L0} + \vec{V}_{C0}| = |V_{L0} - V_{C0}| = |6.12 \text{ V} - 153 \text{ V}| = 147 \text{ V} . \qquad (12.9.13)$$

12.9.3 Resonance

A sinusoidal voltage $V(t) = (200\text{ V})\sin(\omega t)$ is applied to a series RLC circuit with $L = 10.0 \text{ mH}$, $C = 100 \text{ nF}$, and R = 20.0 Ω. Find the following quantities:

(a) the resonant frequency,

(b) the amplitude of the current at resonance,

(c) the quality factor Q_{qual} of the circuit, and

(d) the amplitude of the voltage across the inductor at the resonant frequency.

Solution:

(a) The resonant frequency for the circuit is given by

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{(10.0 \times 10^{-3} \text{ H})(100 \times 10^{-9} \text{ F})}} = 5033 \text{ Hz} . \quad (12.9.14)$$

(b) At resonance, the current is

$$I_0 = \frac{V_0}{R} = \frac{200 \text{ V}}{20.0 \ \Omega} = 10.0 \text{ A} . \qquad (12.9.15)$$

(c) The quality factor Q_{qual} of the circuit is given by

$$Q_{\text{qual}} = \frac{\omega_0 L}{R} = \frac{2\pi(5033\ \text{s}^{-1})(10.0\times10^{-3}\ \text{H})}{(20.0\ \Omega)} = 15.8\,. \qquad (12.9.16)$$

(d) At resonance, the amplitude of the voltage across the inductor is

$$V_{L0} = I_0 X_L = I_0\omega_0 L = (10.0\ \text{A})2\pi(5033\ \text{s}^{-1})(10.0\times10^{-3}\ \text{H}) = 3.16\times10^3\,\text{V}\,. \qquad (12.9.17)$$

12.9.4 *RL* High-Pass Filter

A *RL* high-pass filter (circuit that filters out low-frequency AC currents) can be represented by the circuit in Figure 12.9.2, where *R* is the internal resistance of the inductor.

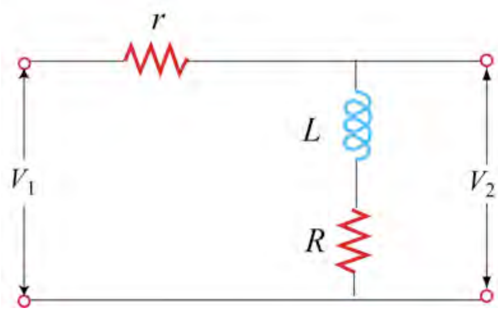


Figure 12.9.2 *RL* filter

(a) Find V_{20}/V_{10}, the ratio of the maximum output voltage V_{20} to the maximum input voltage V_{10}.

(b) Suppose $r = 15.0\ \Omega$, $R = 10\Omega$, and $L = 250\ \text{mH}$. Find the frequency at which $V_{20}/V_{10} = 1/2$.

Solution:

(a) The impedance for the input circuit is $Z_1 = \sqrt{(R+r)^2 + X_L^2}$ where $X_L = \omega L$. The impedance for the output circuit is $Z_2 = \sqrt{R^2 + X_L^2}$. The maximum current is given by

$$I_0 = \frac{V_{10}}{Z_1} = \frac{V_0}{\sqrt{(R+r)^2 + X_L^2}}\,. \qquad (12.9.18)$$

Similarly, the maximum output voltage is related to the output impedance by

$$V_{20} = I_0 Z_2 = I_0\sqrt{R^2 + X_L^2}\,. \tag{12.9.19}$$

This implies

$$\frac{V_{20}}{V_{10}} = \frac{\sqrt{R^2 + X_L^2}}{\sqrt{(R+r)^2 + X_L^2}}\,. \tag{12.9.20}$$

(b) For $V_{20}/V_{10} = 1/2$, we have

$$\frac{R^2 + X_L^2}{(R+r)^2 + X_L^2} = \frac{1}{4} \quad \Rightarrow \quad X_L = \sqrt{\frac{(R+r)^2 - 4R^2}{3}}\,. \tag{12.9.21}$$

Because $X_L = \omega L = 2\pi fL$, the frequency that yields this ratio is

$$f = \frac{X_L}{2\pi L} = \frac{1}{2\pi(0.250\ \text{H})}\sqrt{\frac{(10.0\ \Omega + 15.0\ \Omega)^2 - 4(10.0\ \Omega)^2}{3}} = 5.51\,\text{Hz}\,. \tag{12.9.22}$$

12.9.5 *RLC* Circuit

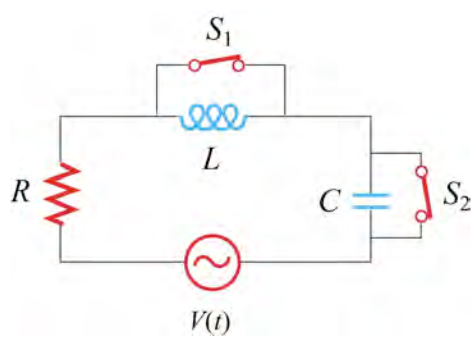


Figure 12.9.3

Consider the circuit shown in Figure 12.9.3. The sinusoidal voltage source is $V(t) = V_0\sin(\omega t)$. If both switches S_1 and S_2 are closed initially, find the following quantities, ignoring the transient effect and assuming that R, L, V_0, and ω are known.

(a) The current $I(t)$ as a function of time.

(b) The average power delivered to the circuit.

(c) The current as a function of time a long time after only S_1 is opened.

(d) The capacitance C if both S_1 and S_2 are opened for a long time, with the current and voltage in phase.

(e) The impedance of the circuit when both S_1 and S_2 are opened.

(f) The maximum energy stored in the capacitor during oscillations.

(g) The maximum energy stored in the inductor during oscillations.

(h) The phase difference between the current and the voltage if the frequency of $V(t)$ is doubled.

(i) The frequency at which the inductive reactance X_L is equal to half the capacitive reactance X_C.

Solutions:

(a) When both switches S_1 and S_2 are closed, the current only goes through the generator and the resistor, so the total impedance of the circuit is R and the current is

$$I_R(t) = \frac{V_0}{R}\sin(\omega t)\,. \tag{12.9.23}$$

(b) The average power is given by

$$\langle P(t)\rangle = \langle I_R(t)V(t)\rangle = \frac{V_0^2}{R}\langle \sin^2(\omega t)\rangle = \frac{V_0^2}{2R}\,. \tag{12.9.24}$$

(c) If only S_1 is opened, after a long time the current will pass through the generator, the resistor and the inductor. For this RL circuit, the impedance becomes

$$Z = \frac{1}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{R^2 + \omega^2 L^2}}\,, \tag{12.9.25}$$

and the phase constant ϕ is

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right). \tag{12.9.26}$$

Thus, the current as a function of time is

$$I(t) = I_0 \sin(\omega t - \phi) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\sin[\omega t - \tan^{-1}(\omega L / R)]\,. \tag{12.9.27}$$

Note that in the limit of vanishing resistance $R=0$, $\phi=\pi/2$, and we recover the expected result for a purely inductive circuit.

(d) If both switches are opened, then this would be a driven RLC circuit, with the phase constant ϕ given by

$$\phi=\tan^{-1}\left(\frac{X_L-X_C}{R}\right)=\tan^{-1}\left(\frac{\omega L-\dfrac{1}{\omega C}}{R}\right). \tag{12.9.28}$$

If the current and the voltage are in phase, then $\phi=0$, implying $\tan\phi=0$. Let the corresponding angular frequency be ω_0; we then obtain

$$\omega_0 L=\frac{1}{\omega_0 C}. \tag{12.9.29}$$

Therefore the capacitance is

$$C=\frac{1}{\omega_0{}^2 L}. \tag{12.9.30}$$

(e) From (d), when both switches are opened, the circuit is at resonance with $X_L=X_C$. Thus, the impedance of the circuit becomes

$$Z=\sqrt{R^2+(X_L-X_C)^2}=R. \tag{12.9.31}$$

(f) The electric energy stored in the capacitor is

$$U_E=\frac{1}{2}CV_C^2=\frac{1}{2}C(IX_C)^2. \tag{12.9.32}$$

It attains maximum when the current is at its maximum I_0,

$$U_{C,\max}=\frac{1}{2}CI_0^2X_C^2=\frac{1}{2}C\left(\frac{V_0}{R}\right)^2\frac{1}{\omega_0{}^2C^2}=\frac{V_0^2L}{2R^2}, \tag{12.9.33}$$

where we have used $\omega_0^2=1/LC$.

(g) The maximum energy stored in the inductor is given by

$$U_{L,\max}=\frac{1}{2}LI_0^2=\frac{LV_0^2}{2R^2}. \tag{12.9.34}$$

(h) If the frequency of the voltage source is doubled, i.e., $\omega = 2\omega_0 = 1/\sqrt{LC}$, then the phase constant becomes

$$\phi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) = \tan^{-1}\left(\frac{(2/\sqrt{LC})L - (\sqrt{LC}/2C)}{R}\right) = \tan^{-1}\left(\frac{3}{2R}\sqrt{\frac{L}{C}}\right). \quad (12.9.35)$$

(i) If the inductive reactance is one-half the capacitive reactance,

$$X_L = \frac{1}{2}X_C \quad \Rightarrow \quad \omega L = \frac{1}{2}\frac{1}{\omega C}. \quad (12.9.36)$$

This occurs when eh angular frequency is

$$\omega = \frac{1}{\sqrt{2LC}} = \frac{\omega_0}{\sqrt{2}}. \quad (12.9.37)$$

12.9.6 *RL* Filter

The circuit shown in Figure 12.9.4 represents a RL filter.

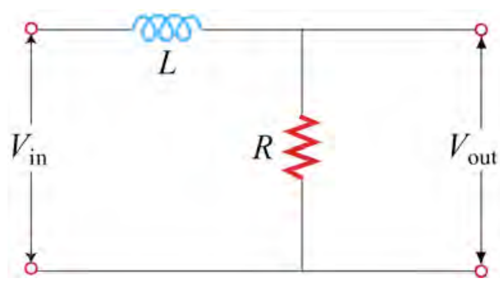


Figure 12.9.4

Let the inductance be L = 400 mH, and the input voltage $V_{in} = (20.0\,\text{V})\sin(\omega t)$, where $\omega = 200$ rad/s .

(a) What is the value of R such that the output voltage lags behind the input voltage by 30.0° ?

(b) Find the ratio of the amplitude of the output and the input voltages. What type of filter is this circuit, high-pass or low-pass?

(c) If the positions of the resistor and the inductor were switched, would the circuit be a high-pass or a low-pass filter?

Solutions:

(a) Because the output voltage V_{out} is measured across the resistor, it is in phase with the current. Therefore the phase difference ϕ between V_{out} and V_{in} is equal to the phase constant and satisfies

$$\tan\phi = \frac{V_L}{V_R} = \frac{IX_L}{IX_R} = \frac{\omega L}{R}. \tag{12.9.38}$$

Thus, we have

$$R = \frac{\omega L}{\tan\phi} = \frac{(200\,\text{rad/s})(0.400\,\text{H})}{\tan 30.0^\circ} = 139\,\Omega. \tag{12.9.39}$$

(b) The ratio is given by

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{V_R}{V_{\text{in}}} = \frac{R}{\sqrt{R^2 + X_L^2}} = \cos\phi = \cos 30.0^\circ = 0.866. \tag{12.9.40}$$

The circuit is a low-pass filter, since the ratio $V_{\text{out}} / V_{\text{in}}$ decreases with increasing ω.

(c) In this case, the circuit diagram is

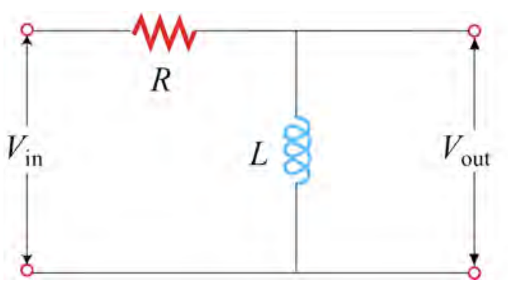


Figure 12.9.5 *RL* high-pass filter

The ratio of the output voltage to the input voltage would be

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{V_L}{V_{\text{in}}} = \frac{X_L}{\sqrt{R^2 + X_L^2}} = \frac{\omega^2 L^2}{\sqrt{R^2 + \omega^2 L^2}} = \left[1 + \left(\frac{R}{\omega L}\right)^2\right]^{-1/2}.$$

The circuit is a high-pass filter, since the ratio $V_{\text{out}} / V_{\text{in}}$ approaches one in the limit as $\omega >> R / L$.

12.10 Conceptual Questions

1. Consider a purely capacitive circuit (a capacitor connected to an AC source).

(a) How does the capacitive reactance change if the driving frequency is doubled? halved?

(b) Are there any times when the capacitor is supplying power to the AC source?

2. If the applied voltage leads the current in a series RLC circuit, is the frequency above or below resonance?

3. Consider the phasor diagram shown in Figure 12.10.1 for a series RLC circuit.

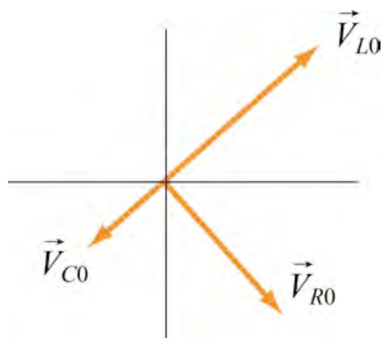


Figure 12.10.1

(a) Is the driving frequency above or below the resonant frequency?

(b) Draw the phasor $\vec{V}_0$ associated with the amplitude of the applied voltage.

(c) Give an estimate of the phase constant ϕ between the applied AC voltage and the current.

4. How does the power factor in a RLC circuit change with resistance R, inductance L, and capacitance C?

5. Can a battery be used as the primary voltage source in a transformer?

6. If the power factor in a RLC circuit is $\cos\phi = 1/2$, can you tell whether the current leading or lagging the voltage? Explain.

12.11 Additional Problems

12.11.1 Reactance of a Capacitor and an Inductor

(a) A $C = 0.5-\mu\text{F}$ capacitor is connected, as shown in Figure 12.11.1(a), to an AC generator $V(t) = V_0 \sin(\omega t)$ with $V_0 = 300\text{ V}$. What is the amplitude I_0 of the resulting alternating current if the angular frequency ω is (i) 100 rad/s, and (ii) 1000 rad/s?

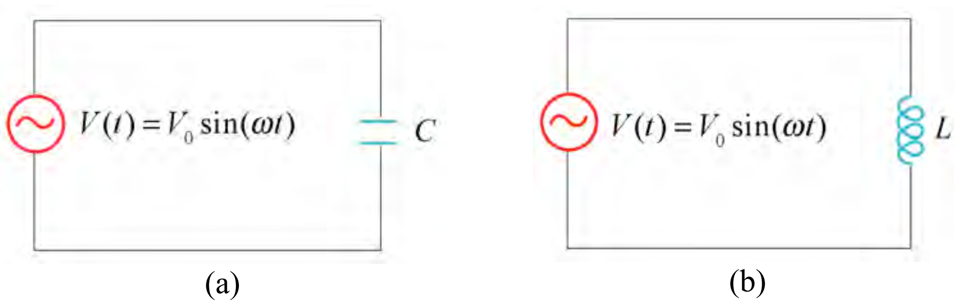


Figure 12.11.1 (a) A purely capacitive circuit, and (b) a purely inductive circuit.

(b) A 45-mH inductor is connected, as shown in Figure 12.10.1(b), to an AC generator $V(t) = V_0 \sin(\omega t)$ with $V_0 = 300\text{ V}$. The inductor has a reactance $X_L = 1300\ \Omega$.

(i) What is the applied angular frequency ω and, (ii) the applied frequency f, in order that $X_L = 1300\ \Omega$?

(iii) What is the amplitude I_0 of the resulting alternating current?

(c) At what frequency f would our 0.5-μF capacitor and our 45-mH inductor have the same reactance? What would this reactance be? How would this frequency compare to the natural resonant frequency of free oscillations if the components were connected as an LC oscillator with zero resistance?

12.11.2 Driven *RLC* Circuit Near Resonance

The circuit shown in Figure 12.11.2 contains an inductor L, a capacitor C, and a resistor R in series with an AC generator, which provides a source of sinusoidally varying emf $V(t) = V_0 \sin(\omega t)$. This emf drives current $I(t) = I_0 \sin(\omega t - \phi)$ through the circuit at angular frequency ω.

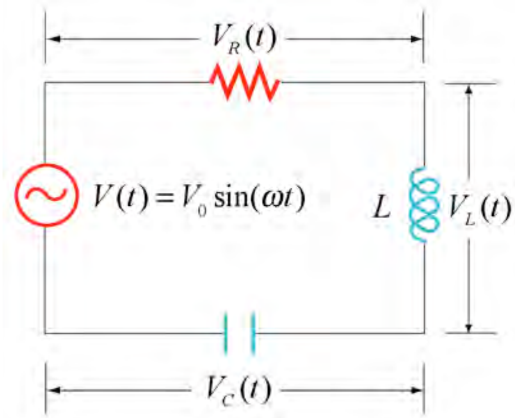


Figure 12.11.2

(a) At what angular frequency ω will the circuit resonate with maximum response, as measured by the amplitude I_0 of the current in the circuit? What is the value of the maximum current amplitude $I_{\max}$?

(b) What is the value of the phase constant ϕ (the phase difference between $V(t)$ and $I(t)$) at this resonant angular frequency?

(c) Suppose the angular frequency ω is increased from the resonance value until the amplitude I_0 of the current decreases from $I_{\max}$ to $I_{\max}/\sqrt{2}$. What is new value of the phase difference ϕ between the emf and the current? Does the current lead or lag the emf?

12.11.3 *RC* Circuit

A series RC circuit with $R = 4.0 \times 10^3\,\Omega$ and $C = 0.40\ \mu\text{F}$ is connected to an AC voltage source $V(t) = (100\ \text{V})\sin(\omega t)$, with $\omega = 200\ \text{rad/s}$.

(a) What is the rms current in the circuit?

(b) What is the phase difference between the voltage and the current?

(c) Find the power dissipated in the circuit.

(d) Find the voltage drop both across the resistor and the capacitor.

12.11.4 Black Box

An AC voltage source is connected to a "black box" which contains a circuit, as shown in Figure 12.11.3.

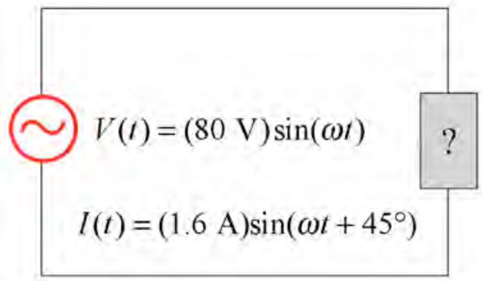


Figure 12.11.3 A "black box" connected to an AC voltage source.

The elements in the circuit and their arrangement, however, are unknown. Measurements outside the black box provide the following information: $V(t) = (80\text{ V})\sin(\omega t)$, and $I(t) = (1.6\text{ A})\sin(\omega t + 45°)$.

(a) Does the current lead or lag the voltage?

(b) Is the circuit in the black box largely capacitive or inductive?

(c) Is the circuit in the black box at resonance?

(d) What is the power factor?

(e) Does the box contain a resistor? A capacitor? An inductor?

(f) Compute the average power delivered to the black box by the AC source.

12.11.5 Parallel *RL* Circuit

Consider the parallel RL circuit shown in Figure 12.11.4.

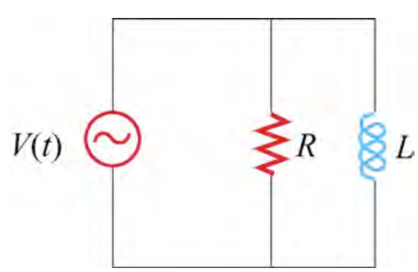


Figure 12.11.4 Parallel RL circuit

The AC voltage source is $V(t) = V_0 \sin(\omega t)$.

(a) Find the current across the resistor.

(b) Find the current across the inductor.

(c) What is the magnitude of the total current?

(d) Find the impedance of the circuit.

(e) What is the phase difference between the current and the voltage?

12.11.6 Parallel *RC* Circuit

Consider the parallel RC circuit shown in Figure 12.11.5.

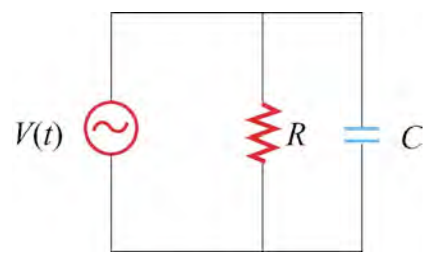


Figure 12.11.5 Parallel RC circuit

The AC voltage source is $V(t) = V_0 \sin(\omega t)$.

(a) Find the current across the resistor.

(b) Find the current across the capacitor.

(c) What is the magnitude of the total current?

(d) Find the impedance of the circuit.

(e) What is the phase difference between the current and the voltage?

12.11.7 Power Dissipation

A series RLC circuit with $R = 10.0\ \Omega$, $L = 400\ \text{mH}$, and $C = 2.0\ \mu\text{F}$ is connected to an AC voltage source $V(t) = V_0 \sin(\omega t)$ that has amplitude $V_0 = 100\ \text{V}$.

(a) What is the resonant angular frequency ω_0?

(b) Find the rms current at resonance.

(c) Let the driving angular frequency be $\omega = 4000\text{ rad/s}$. Compute X_C, X_L, Z, and ϕ.

12.11.8 FM Antenna

An FM antenna circuit (shown in Figure 12.11.6) has an inductance $L = 10^{-6}\text{ H}$, a capacitance $C = 10^{-12}\text{ F}$, and a resistance $R = 100\,\Omega$. A radio signal induces a sinusoidally alternating emf in the antenna with amplitude 10^{-5} V.

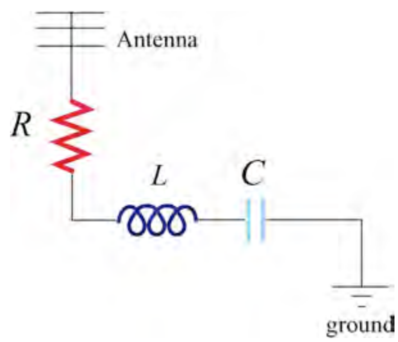


Figure 12.11.6

(a) For what angular frequency ω_0 (radians/sec) of the incoming waves will the circuit be "in tune"-- that is, for what ω_0 will the current in the circuit be a maximum.

(b) What is the quality factor Q_{qual} of the resonance?

(c) Assuming that the incoming wave is "in tune," what will be the amplitude of the current in the circuit at this "in tune" angular frequency.

(d) What is the amplitude of the potential difference across the capacitor at this "in tune" angular frequency?

12.11.9 Driven *RLC* Circuit

Suppose you want a series *RLC* circuit to tune to your favorite FM radio station that broadcasts at a frequency of $89.7\,\text{MHz}$. You would like to avoid the station that broadcasts at $89.5\,\text{MHz}$. In order to achieve this, for a given input voltage signal from your antenna, you want the width of your resonance to be narrow enough at $89.7\,\text{MHz}$ such that the current flowing in your circuit will be 10^{-2} times less at $89.5\,\text{MHz}$ than at

$89.7\,\text{MHz}$. You cannot avoid having a resistance of $R = 0.1\Omega$, and practical considerations also dictate that you use the minimum L possible.

(a) In terms of your circuit parameters, L, R and C, what is the amplitude of your current in your circuit as a function of the angular frequency of the input signal?

(b) What is the angular frequency of the input signal at the desired resonance?

(c) What values of L and C must you use?

(d) What is the quality factor for this resonance?

(e) Show that at resonance, the ratio of the amplitude of the voltage across the inductor with the driving signal amplitude is the quality of the resonance.

(f) Show that at resonance the ratio of the amplitude of the voltage across the capacitor with the driving signal amplitude is the quality of the resonance.

(g) What is the time-averaged power at resonance that the signal delivers to the circuit?

(h) What is the phase constant for the input signal at $89.5\,\text{MHz}$?

(i) What is the time-averaged power for the input signal at $89.5\,\text{MHz}$?

(j) Is the circuit capacitive or inductive at $89.5\,\text{MHz}$?