Chapter 11

Inductance and Magnetic Energy

Inductance and Magnetic Energy

11.1 Mutual Inductance

Suppose two coils are placed near each other, as shown in Figure 11.1.1

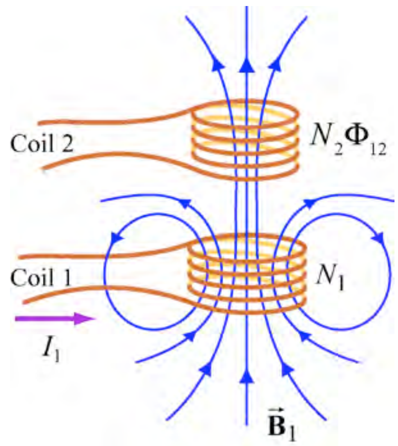


Figure 11.1.1 Changing current in coil 1 produces changing magnetic flux in coil 2.

The first coil has N_1 turns and carries a current I_1 which gives rise to a magnetic field $\vec{\mathbf{B}}_1$. The second coil has N_2 turns. Because the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let Φ_{12} denote the magnetic flux through one turn of coil 2 due to I_1. Now, by varying I_1 with time, there will be an induced emf associated with the changing magnetic flux in the second coil:

$$\varepsilon_{12} = -N_2 \frac{d\Phi_{12}}{dt} = -\frac{d}{dt} \iint_{\text{coil 2}} \vec{\mathbf{B}}_1 \cdot d\vec{\mathbf{A}}_2 \,. \tag{11.1.1}$$

The time rate of change of magnetic flux Φ_{12} in coil 2 is proportional to the time rate of change of the current in coil 1:

$$N_2 \frac{d\Phi_{12}}{dt} = M_{12} \frac{dI_1}{dt} \,, \tag{11.1.2}$$

where the proportionality constant M_{12} is called the **mutual inductance**. It can also be written as

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1} \,. \tag{11.1.3}$$

The SI unit for inductance is the henry [H]:

$$1\ \text{henry} = 1\ \text{H} = 1\ \text{T}\cdot\text{m}^2/\text{A}\,. \qquad (11.1.4)$$

We shall see that the mutual inductance M_{12} depends only on the geometrical properties of the two coils such as the number of turns and the radii of the two coils.

In a similar manner, suppose instead there is a current I_2 in the second coil and it is varying with time (Figure 11.1.2). Then the induced emf in coil 1 becomes

$$\varepsilon_{21} = -N_1 \frac{d\Phi_{21}}{dt} = -\frac{d}{dt}\iint_{\text{coil 1}} \vec{\mathbf{B}}_2 \cdot d\vec{\mathbf{A}}_1\,, \qquad (11.1.5)$$

and a current is induced in coil 1.

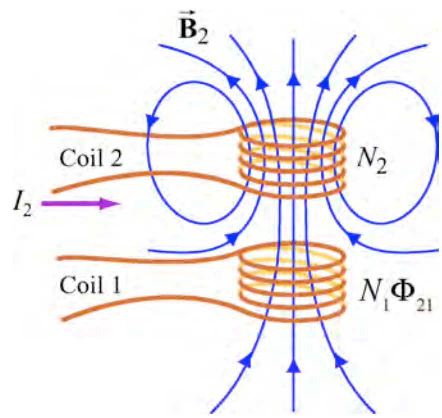


Figure 11.1.2 Changing current in coil 2 produces changing magnetic flux in coil 1.

This changing flux in coil 1 is proportional to the changing current in coil 2,

$$N_1 \frac{d\Phi_{21}}{dt} = M_{21} \frac{dI_2}{dt}\,, \qquad (11.1.6)$$

where the proportionality constant M_{21} is another mutual inductance and can be written as

$$M_{21} = \frac{N_1\Phi_{21}}{I_2}\,. \qquad (11.1.7)$$

The *mutual inductance reciprocity theorem* states that the constants are equal

$$M_{12} = M_{21} \equiv M\,. \qquad (11.1.8)$$

We shall not prove this theorem. It's left as a difficult exercise for the reader to prove this result using Ampere's law and the Biot-Savart law.

Example 11.1 Mutual Inductance of Two Concentric Co-planar Loops

Consider two single-turn co-planar, concentric coils of radii R_1 and R_2, with $R_1 >> R_2$, as shown in Figure 11.1.3. What is the mutual inductance between the two loops?

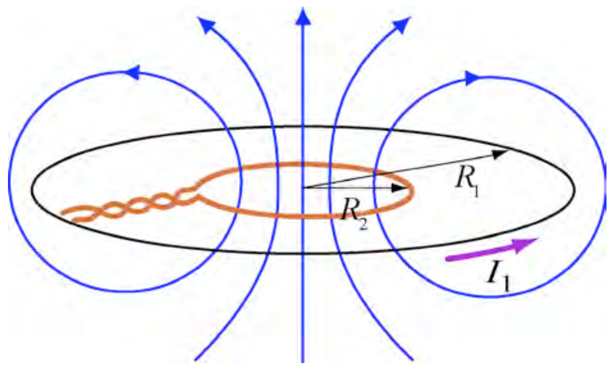


Figure 11.1.3 Two concentric current loops

Solution: The mutual inductance can be computed as follows. Using Eq. (9.1.15) of Chapter 9, we see that the magnitude of the magnetic field at the center of the ring due to I_1 in the outer coil is given by

$$B_1 = \frac{\mu_0 I_1}{2R_1}. \tag{11.1.9}$$

Because $R_1 >> R_2$, we approximate the magnetic field through the entire inner coil by B_1. Hence, the flux through the second (inner) coil is

$$\Phi_{12} = B_1 A_2 = \frac{\mu_0 I_1}{2R_1}\pi R_2^2 = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}. \tag{11.1.10}$$

Thus, the mutual inductance is given by

$$M = \frac{\Phi_{12}}{I_1} = \frac{\mu_0 \pi R_2^2}{2R_1} \tag{11.1.11}$$

The result shows that M depends only on the geometrical factors, R_1 and R_2, and is independent of the current I_1 in the coil.

11.2 Self-Inductance

Consider again a coil consisting of N turns and carrying current I in the counterclockwise direction, as shown in Figure 11.2.1. If the current is steady, then the magnetic flux through the loop will remain constant. However, suppose the current I changes with time, then according to Faraday's law, an induced emf will arise to oppose the change. The induced current will flow clockwise if $dI/dt > 0$, and counterclockwise if $dI/dt < 0$. The property of the loop in which its own magnetic field opposes any change in current is called **self-inductance**, and the emf generated is called the self-induced emf or back emf, which we denote as ε_L. The self-inductance may arise from a coil and the rest of the circuit, especially the connecting wires.

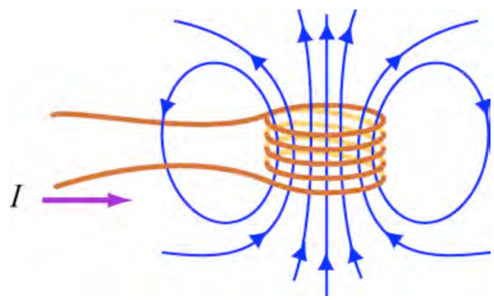


Figure 11.2.1 Magnetic flux through a coil

Mathematically, the self-induced emf can be written as

$$\varepsilon_L = -N\frac{d\Phi_{B,turn}}{dt} = -N\frac{d}{dt}\iint_{turn}\vec{\mathbf{B}}\cdot d\vec{\mathbf{A}}. \tag{11.2.1}$$

Because the flux is proportional to the current I, we can also express this relationship by

$$\varepsilon_L = -L\frac{dI}{dt}. \tag{11.2.2}$$

where the constant L is called the *self-inductance*. The two expressions can be combined to yield

$$L = \frac{N\Phi_B}{I}. \tag{11.2.3}$$

Physically, the self-inductance L is a measure of an inductor's "resistance" to the change of current; the larger the value of L, the lower the rate of change of current.

Example 11.2 Self-Inductance of a Solenoid

Compute the self-inductance of a solenoid with N turns, length l, and radius R with a current I flowing through each turn, as shown in Figure 11.2.2.

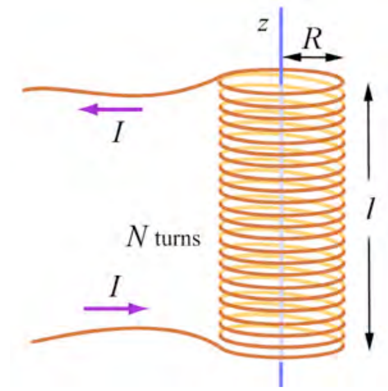


Figure 11.2.2 Solenoid

Solution: Ignoring edge effects and applying Ampere's law, the magnetic field inside a solenoid is given by Eq. (9.4.3)

$$\vec{\mathbf{B}} = \frac{\mu_0 NI}{l}\hat{\mathbf{k}} = \mu_0 nI\,\hat{\mathbf{k}}\,, \tag{11.2.4}$$

where $n = N/l$ is the number of turns per unit length. The magnetic flux through each turn is

$$\Phi_B = BA = \mu_0 nI\cdot(\pi R^2) = \mu_0 nI\pi R^2\,. \tag{11.2.5}$$

Thus, the self-inductance is

$$L = \frac{N\Phi_B}{I} = \mu_0 n^2 \pi R^2 l\,. \tag{11.2.6}$$

We see that L depends only on the geometrical factors (n, R and l) and is independent of the current I.

Example 11.3 Self-Inductance of a Toroid

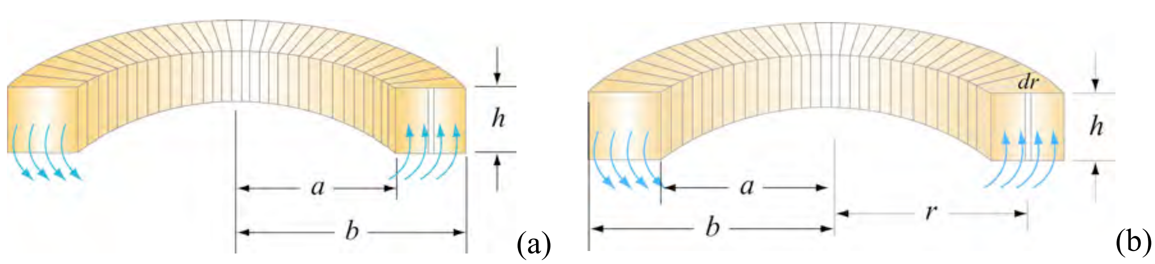


Figure 11.2.3 A toroid with N turns

Calculate the self-inductance L of a toroid, which consists of N turns and has a rectangular cross section, with inner radius a, outer radius b, and height h, as shown in Figure 11.2.3(a).

Solution: According to Ampere's law (Example 9.5),

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \oint B\,ds = B\oint ds = B(2\pi r) = \mu_0 NI\,. \tag{11.2.7}$$

The magnitude of the magnetic field inside the torus is given by

$$B = \frac{\mu_0 NI}{2\pi r}\,. \tag{11.2.8}$$

The magnetic flux through one turn of the toroid may be obtained by integrating over the rectangular cross section, with $dA = h\,dr$ as the differential area element (Figure 11.2.3b),

$$\Phi_B = \iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \int_a^b \left(\frac{\mu_0 NI}{2\pi r}\right) h\,dr = \frac{\mu_0 NIh}{2\pi} \ln\left(\frac{b}{a}\right). \tag{11.2.9}$$

The total flux is $N\Phi_B$. Therefore, the self-inductance is

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right). \tag{11.2.10}$$

Again, the self-inductance L depends only on the geometrical factors. Let's consider the situation where $a >> b-a$. In this limit, the logarithmic term in the equation above may be expanded as

$$\ln\left(\frac{b}{a}\right) = \ln\left(1 + \frac{b-a}{a}\right) \approx \frac{b-a}{a}\,, \tag{11.2.11}$$

and the self-inductance becomes

$$L \approx \frac{\mu_0 N^2 h}{2\pi} \cdot \frac{b-a}{a} = \frac{\mu_0 N^2 A}{2\pi a} = \frac{\mu_0 N^2 A}{l}\,, \tag{11.2.12}$$

where $A = h(b-a)$ is the cross-sectional area, and $l = 2\pi a$. We see that the self-inductance of the toroid in this limit has the same form as that of a solenoid.

Example 11.4 Mutual Inductance of a Coil Wrapped Around a Solenoid

A long solenoid with length l and a cross-sectional area A consists of N_1 turns of wire. An insulated coil of N_2 turns is wrapped around it, as shown in Figure 11.2.4.

(a) Calculate the mutual inductance M, assuming that all the flux from the solenoid passes through the outer coil.

(b) Relate the mutual inductance M to the self-inductances L_1 and L_2 of the solenoid and the coil.

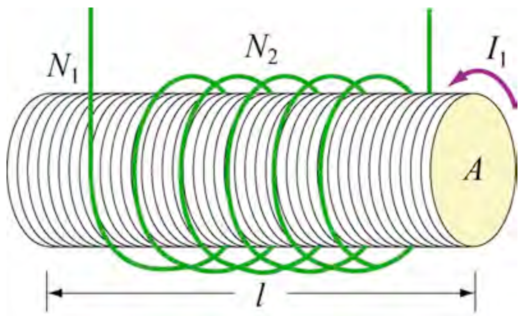


Figure 11.2.4 A coil wrapped around a solenoid

Solutions:

(a) The magnetic flux through each turn of the outer coil due to the solenoid is

$$\Phi_{12} = BA = \frac{\mu_0 N_1 I_1}{l} A. \tag{11.2.13}$$

where $B = \mu_0 N_1 I_1 / l$ is the uniform magnetic field inside the solenoid. Thus, the mutual inductance is

$$M = \frac{N_2 \Phi_{12}}{I_1} = \frac{\mu_0 N_1 N_2 A}{l}. \tag{11.2.14}$$

(b) From Example 11.2, we see that the self-inductance of the solenoid with N_1 turns is given by

$$L_1 = \frac{N_1 \Phi_{11}}{I_1} = \frac{\mu_0 N_1^2 A}{l}, \tag{11.2.15}$$

where Φ_{11} is the magnetic flux through one turn of the inner solenoid due to the magnetic field produced by I_1. Similarly, we have $L_2 = \mu_0 N_2^2 A / l$ for the outer coil. In terms of L_1 and L_2, the mutual inductance can be written as

$$M = \sqrt{L_1 L_2} \,. \tag{11.2.16}$$

More generally the mutual inductance is given by

$$M = k\sqrt{L_1 L_2}\,, \qquad 0 \le k \le 1\,, \tag{11.2.17}$$

where k is the *coupling coefficient*. In our example, we have $k = 1$, which means that all of the magnetic flux produced by the solenoid passes through the outer coil, and vice versa, in this idealization.

11.3 Energy Stored in Magnetic Fields

Because an inductor in a circuit serves to oppose any change in the current through it, work must be done by an external source such as a battery in order to establish a current in the inductor. From the work-energy theorem, we conclude that energy can be stored in an inductor. The role played by an inductor in the magnetic case is analogous to that of a capacitor in the electric case.

The power, or rate at which an external emf ε_{ext} works to overcome the self-induced emf ε_L and pass current I in the inductor is

$$P_L = \frac{dW_{\text{ext}}}{dt} = I\varepsilon_{\text{ext}} \,. \tag{11.3.1}$$

If only the external emf and the inductor are present, then $\varepsilon_{\text{ext}} = -\varepsilon_L$ which implies that

$$P_L = \frac{dW_{\text{ext}}}{dt} = -I\varepsilon_L = +IL\frac{dI}{dt} \,. \tag{11.3.2}$$

If the current is increasing with $dI/dt > 0$, then $P > 0$, which means that the external source is doing positive work to transfer energy to the inductor. Thus, the internal energy U_B of the inductor is increased. On the other hand, if the current is decreasing with $dI/dt < 0$, we then have $P < 0$. In this case, the external source takes energy away from the inductor, causing its internal energy to decrease. The total work done by the external source to increase the current form zero to I is then

$$W_{\text{ext}} = \int dW_{\text{ext}} = \int_0^I LI'\,dI' = \frac{1}{2}LI^2 \,. \tag{11.3.3}$$

This is equal to the magnetic energy stored in the inductor,

$$U_B = \frac{1}{2}LI^2 . \tag{11.3.4}$$

The above expression is analogous to the electric energy stored in a capacitor,

$$U_E = \frac{1}{2}\frac{Q^2}{C} . \tag{11.3.5}$$

From an energy perspective there is an important distinction between an inductor and a resistor. Whenever a current I goes through a resistor, energy flows into the resistor and dissipates in the form of heat regardless of whether I is steady or time-dependent (recall that power dissipated in a resistor is $P_R = IV_R = I^2 R$). Energy flows into an ideal inductor only when the current is varying with $dI/dt > 0$. The energy is not dissipated but stored there; it is released later when the current decreases with $dI/dt < 0$. If the current that passes through the inductor is steady, then there is no change in energy since $P_L = LI(dI/dt) = 0$.

Example 11.5 Energy Stored in a Solenoid

A long solenoid with length l and a radius R consists of N turns of wire. A current I passes through the coil. Find the energy stored in the system.

Solution: Using Eqs. (11.2.6) and (11.3.4), we readily obtain

$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 I^2 \pi R^2 l . \tag{11.3.6}$$

The result can be expressed in terms of the magnetic field strength $B = \mu_0 nI$,

$$U_B = \frac{1}{2\mu_0}(\mu_0 nI)^2(\pi R^2 l) = \frac{B^2}{2\mu_0}(\pi R^2 l) . \tag{11.3.7}$$

Because $\pi R^2 l$ is the volume within the solenoid, and the magnetic field inside is uniform, the **magnetic energy density**, or the energy per unit volume of the magnetic field is given by

$$u_B = \frac{B^2}{2\mu_0} \tag{11.3.8}$$

The above expression holds true even when the magnetic field is non-uniform. The result can be compared with the energy density associated with an electric field,

$$u_E = \frac{1}{2}\varepsilon_0 E^2. \tag{11.3.9}$$

11.3.1 Creating and Destroying Magnetic Energy Animation

Let's consider the process involved in creating magnetic energy. Here we discuss this process qualitatively. A quantitative calculation is given in Section 13.6.2. Figure 11.3.1 shows the process by which an external agent(s) creates magnetic energy. Suppose we have five rings that carry a number of free positive charges that are not moving. Because there is no current, there is no magnetic field. Suppose a set of external agents come along (one for each charge) and simultaneously spin up the charges counterclockwise as seen from above, at the same time and at the same rate, in a manner that has been pre-arranged. Once the charges on the rings start to move, there is a magnetic field in the space between the rings, mostly parallel to their common axis, which is stronger inside the rings than outside. This is the solenoid configuration (see Section 9.4).

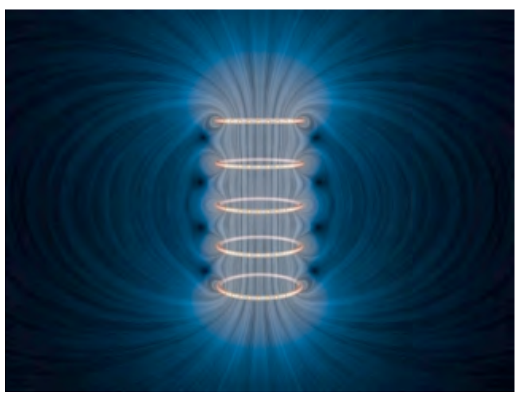


Figure 11.3.1 Creating (http://youtu.be/GI2Prj4CGZI) and destroying (http://youtu.be/iesoHVfIg6I) magnetic field energy.

As the magnetic flux through the rings grows, Faraday's law of induction tells us that there is an electric field induced by the time-changing magnetic field that is circulating clockwise as seen from above. The force on the charges due to this induced electric field is thus opposite the direction the external agents are trying to spin the rings up (counterclockwise), and thus the agents have to do additional work to spin up the charges because of their charge. This is the source of the energy that is appearing in the magnetic field between the rings — the work done by the agents against the *back emf*.

Over the course of the "create" animation associated with Figure 11.3.1, the agents moving the charges to a higher speed against the induced electric field are continually doing work. The electromagnetic energy is being created at the place where they are doing work (the path along which the charges move) and that electromagnetic energy flows primarily inward, but also outward. The direction of the flow of this energy is shown by the animated texture patterns in Figure 11.3.1. This is the electromagnetic energy flow that increases the strength of the magnetic field in the space between the rings as each positive charge is accelerated to a higher and higher speed. When the external agents have spun the charges to a pre-determined speed, they stop the acceleration. The charges then move at a constant speed, with a constant field inside the solenoid, and zero induced electric field, in accordance with Faraday's law of induction.

We also have an animation of the "destroy" process linked to Figure 11.3.1. This process proceeds as follows. Our set of external agents now simultaneously starts to spin down the moving charges (which are still moving counterclockwise as seen from above), at the same time and at the same rate, in a manner that has been pre-arranged. Once the charges on the rings start to decelerate, the magnetic field in the space between the rings starts to decrease in magnitude. As the magnetic flux through the rings decreases, Faraday's law tells us that there is now an electric field induced by the time-changing magnetic field that is circulating counterclockwise as seen from above. The force on the charges due to this electric field is thus in the same direction as the motion of the charges. In this situation the agents have work done on them as they try to spin the charges down.

Over the course of the "destroy" animation associated with Figure 11.3.1, the strength of the magnetic field decreases, and this energy flows from the field back to the path along which the charges move, and is now being provided *to* the agents trying to spin down the moving charges. The energy provided to those agents as they destroy the magnetic field is exactly the amount of energy that they put into creating the magnetic field in the first place, neglecting radiative losses (such losses are small if we move the charges at speeds small compared to the speed of light). This is a totally reversible process if we neglect such losses. That is, the amount of energy the agents put into creating the magnetic field is exactly returned to the agents as the field is destroyed.

There is one final point to be made. Whenever electromagnetic energy is being created, an electric charge is moving (or being moved) against an electric field ($q\vec{\mathbf{v}}\cdot\vec{\mathbf{E}}<0$). Whenever electromagnetic energy is being destroyed, an electric charge is moving (or being moved) along an electric field ($q\vec{\mathbf{v}}\cdot\vec{\mathbf{E}}>0$). This is the same rule we saw above when we were creating and destroying electric energy above.

11.4 *RL* Circuits

11.4.1 Self-Inductance and the Faraday's Law

The addition of time-changing magnetic fields to simple circuits means that the closed line integral of the electric field around a circuit is no longer zero (Chapter 10.3). Instead, we have, for any open surface

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\iint \frac{\partial}{\partial t}\vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}\,. \tag{11.4.1}$$

Any circuit in which the current changes with time will have time-changing magnetic fields, and therefore associated induced electric fields, which are due to the time changing currents, not to the time changing magnetic field (association is not causation). How do we solve simple circuits taking such effects into account? We discuss here a consistent way to understand the consequences of introducing time-changing magnetic fields into circuit theory--that is, self-inductance.

As soon as we introduce time-changing currents, and thus time changing magnetic fields, the electric potential difference between two points in our circuit is no longer well defined. When the line integral of the electric field around a closed loop is no longer zero, the potential difference between any two points a and b, is no longer independent of the path used to get from a to b. That is, the electric field is no longer an electrostatic (conservative) field, and the electric potential is no longer an appropriate concept (that is, $\vec{\mathbf{E}}$ can no longer be written as the negative gradient of a scalar potential). However, we can still write down in a straightforward fashion the differential equation for the current $I(t)$ that determines the time-behavior of the current in the circuit.

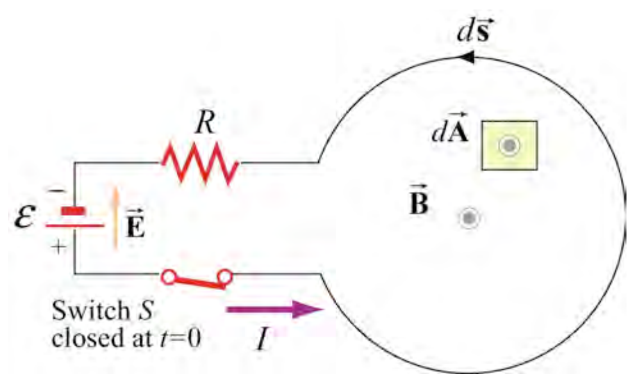


Figure 11.4.1 One-loop inductor circuit

To show how to do this, consider the circuit shown in Figure 11.4.1. We have a battery, a resistor, a switch S that is closed at $t = 0$, and a "one-loop inductor." It will become clear what the consequences of this "inductance" are as we proceed. For $t > 0$, current is in the direction shown (from the positive terminal of the battery to the negative, as usual). What is the equation that governs the behavior of our current $I(t)$ for $t > 0$?

To investigate this, apply Faraday's law to the open surface bounded by our circuit, where we take $d\vec{\mathbf{A}} = dA\,\hat{\mathbf{n}}$ pointing out of the plane of the Figure 11.4.1, and $d\vec{\mathbf{s}}$ is counterclockwise. First, we would like to evaluate the left-hand-side of Eq. (11.4.1), the integral of the electric field around this circuit. There is an electric field in the battery, directed from the positive terminal to the negative terminal, and when we go through the battery in the direction of $d\vec{\mathbf{s}}$ that we have chosen, we are moving against that electric field, so the contribution of the battery to our integral is negative and equal to the negative of the emf provided by the battery,

$$\int_{battery} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\varepsilon .$$

Then, there is an electric field in the resistor, in the direction of the current, so when we move through the resistor in that direction, the contribution to our integral is positive,

$$\int_{\substack{external\\circuit}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = IR \ .$$

What about when we move through our one-loop inductor? There is **no** electric field in this loop if the resistance of the wire making up the loop is zero. Thus, going around the closed loop *counterclockwise* in the direction of the current, we have

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\varepsilon + IR . \qquad (11.4.2)$$

What is the right-hand-side of Eq. (11.4.1)? Because we have assumed in this section that the circuit is not moving, we can take the partial with respect to time outside of the surface integral and then we simply have the time derivative the magnetic flux through the loop. What is the magnetic flux through the open surface? First of all, we arrange the geometry so that the part of the circuit that includes the battery, the switch, and the resistor makes only a small contribution to Φ_B as compared to the (much larger in area) part of the open surface that constitutes our "one-loop inductor". Second, we know that the sign of the magnetic flux is positive in that part of the surface, because current in the counterclockwise direction will produce a magnetic field $\vec{\mathbf{B}}$ pointing out of the plane of Figure 11.4.1, which is the same direction we have assumed for $d\vec{\mathbf{A}}$, so that $\vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$ is positive. Note that our magnetic field here is the *self*-magnetic field—that is the magnetic field produced by the current in the circuit, and not by currents external to this circuit.

We also know that at any point in space, $\vec{\mathbf{B}}$ is proportional to the current *I*, since it can be computed from the Biot-Savart Law, that is,

$$\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \frac{\mu_o I(t)}{4\pi} \oint \frac{d\vec{\mathbf{s}}' \times \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right)}{\left|\left(\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right)\right|^3} \qquad (11.4.3)$$

You may immediately object that the Biot-Savart Law is only good in time-independent situations, but in fact, as long as the current is varying on time scales *T* long compared to the speed of light travel time across the circuit and we are within a distance *cT* of the currents, then (11.4.3) is an excellent approximation to the time dependent magnet field. If we look at (11.4.3), although for a general point in space it involves a very complicated integral over the circuit, it is clear that $\vec{\mathbf{B}}(\vec{\mathbf{r}},t)$ is everywhere proportional to $I(t)$. That is, if we double the current, $\vec{\mathbf{B}}$ at any point in space will also double. It then follows that the magnetic flux itself must be proportional to I, because it is the surface integral of $\vec{\mathbf{B}}$, and $\vec{\mathbf{B}}$ is everywhere proportional to I. That is,

$$\begin{aligned}\Phi_B(t) &= \int_{S(t)} \vec{\mathbf{B}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}}da = \int_S \left\{ \frac{\mu_0 I(t)}{4\pi} \oint \frac{d\vec{\mathbf{s}}' \times (\vec{\mathbf{r}}-\vec{\mathbf{r}}')}{\left|(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\right|^3} \right\}\cdot\hat{\mathbf{n}}\,da \\ &= I(t)\int_S \left\{ \frac{\mu_0}{4\pi} \oint \frac{d\vec{\mathbf{s}}' \times (\vec{\mathbf{r}}-\vec{\mathbf{r}}')}{\left|(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\right|^3} \right\}\cdot\hat{\mathbf{n}}\,da.\end{aligned} \tag{11.4.4}$$

Therefore we can define the self-inductance L by

$$\Phi_B(t) = LI(t) \tag{11.4.5}$$

where

$$L = \int_S \left\{ \frac{\mu_0}{4\pi} \oint \frac{d\vec{\mathbf{s}}' \times (\vec{\mathbf{r}}-\vec{\mathbf{r}}')}{\left|(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\right|^3} \right\}\cdot\hat{\mathbf{n}}\,da\,. \tag{11.4.6}$$

So the magnetic flux is a constant L times the current. Note that L is a constant in the sense that it stays the same as long as we do not change the geometry of the circuit. If we change the geometry of the circuit (for example we halve the radius of the circle in our Figure 11.4.1), we will change L, but for a given geometry, L does not change. Even though it may be terrifically difficult to do the integrals in Eq. (11.4.6), once we have done it for a given circuit geometry we know L, and L is a constant for that geometry. The quantity L is called the *self-inductance* of the circuit, or simply the inductance. From the definition in (11.4.6), you can show that the dimensions of L are μ_0 times a length.

Regardless of how hard or easy it is to compute L, it is a constant for a given circuit geometry and now we can write down the equation that governs the time evolution of I. If $\Phi_B(t) = LI(t)$, then $d\Phi_B(t)/dt = LdI(t)/dt$, and Eq. (11.4.1) becomes

$$-\varepsilon + IR = -L\frac{dI}{dt}\,. \tag{11.4.7}$$

If we divide Eq. (11.4.7) by L and rearrange terms, we find that the equation that determines the time dependence of I is

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{\varepsilon}{L}. \tag{11.4.8}$$

We shall explore the solution to this equation in Example 11.6.1.

11.4.2 Kirchhoff's Loop Rule Modified for Inductors: a Warning

We can write the governing equation for $I(t)$ from above as

$$\sum_i \Delta V_i=\varepsilon-IR-L\frac{dI}{dt}=0 \tag{11.4.9}$$

where we have now cast it in a form that "looks like" a version of Kirchhoff's Second Law, a rule that is often quoted in elementary electromagnetism texts. Kirchhoff's Second Law states that the sum of the potential drops around a circuit is zero. In a circuit with no inductance, this is just a statement that the line integral of the electric field around the circuit is zero, which is certainly true if there is no time variation. However, in circuits with currents that vary in time, this "Law" is no longer true.

Unfortunately, many elementary texts choose to approach circuits with inductance by preserving "Kirchhoff's Second Law", or the loop theorem, by specifying that if the inductor is traversed in the direction of the current, the "potential drop" across an inductor is $-LdI(t)/dt$. Use of this formalism will give the correct equations. However, the continued use of Kirchhoff's Second Law with inductors is misleading at best, for the following reasons.

The continued use of Kirchhoff's Second Law in this way gives the right equations, but it confuses the physics. In particular, saying that there is a "potential drop" across the inductor of $-LdI(t)/dt$ implies that there is an electric field in the inductor such that the integral of $\vec{\mathbf{E}}$ through the inductor is equal to $-LdI(t)/dt$. **This is not always, or even usually, true**. For example, suppose in our "one-loop" inductor (Figure 11.4.1) that the wire making up the loop has negligible resistance compared to the resistance R. The integral of $\vec{\mathbf{E}}$ through our "one-loop" inductor above is then **very small**, **NOT** $-LdI(t)/dt$. Why is it very small? Well, to repeat our assertion above

For a single loop circuit, the current *I* is to an good approximation the same in all parts of the circuit

This is just as valid in a circuit with inductance. Again, although the current may start out at $t=0$ unequal in different parts of the circuit, those inequalities imply that charge is piling up somewhere. The accumulating charge at the pile-up will quickly produce an electric field, and this *electric field is always in the sense so as to smooth out the inequalities in the current*. In this particular case, if the conductivity of the wires making up our one-loop inductor is very large, then there will be a very small electric field in those wires, because it takes only a small electric field to drive any current you need. The amount of current needed is determined in part by the larger resistance in other parts of the circuit, and it is the charge accumulation at the ends of those low conductivity resistors that cancel out the field in the inductor and enhance it in the resistor, maintaining a constant current in the circuit.

One final point may confuse the issue even further. If you have ever put the probes of a voltmeter across the terminals of an inductor (with very small resistance) in a circuit, what you measured on the meter of the voltmeter was a "voltage drop" of $-LdI(t)/dt$. But that is not because there is an electric field in the inductor! It is because putting the voltmeter in the circuit will result in a time changing magnetic flux through the voltmeter circuit, consisting of the inductor, the voltmeter leads, and the large internal resistor in the voltmeter. A current will flow in the voltmeter circuit because there will be an electric field in the large internal resistance of the voltmeter, with a potential drop across *that* resistor of $-LdI(t)/dt$, by Faraday's Law applied to the voltmeter circuit, and that is what the voltmeter will read. The voltmeter as usual gives you a measure of the potential drop across its own internal resistance, but this is *not* a measure of the potential drop across the inductor. It is a measure of the time rate of change of magnetic flux in the voltmeter circuit! As before, there is only a very small electric field in the inductor if it has a very small resistance compared to other resistances in the circuit.

11.4.3 Example Voltmeter Readings with Time Changing Magnetic Fields

We can think of a voltmeter as a device that registers the line integral $\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ along a path from the clip of its (+) lead to the clip of its (-) lead. Part of the path lies inside the voltmeter itself. The path may also be part of a loop, which is completed by some external path from the (-) clip to the (+) clip. With that in mind, consider the arrangement in the Figure 11.4.2.

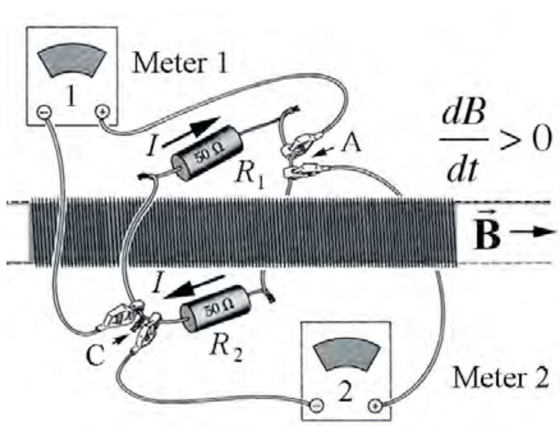


Figure 11.4.2 Two voltmeters and a time changing magnetic field inside a solenoid

The solenoid is so long that its external magnetic field is negligible. Its cross section is 20 cm^2 in area, and the field inside is to the right and increasing at the rate of $10^{-2}\text{ T}\cdot\text{s}^{-1}$. Two identical voltmeters are connected as shown to points A and C on the loop, which encloses the solenoid and contains the two 50-ohm resistors. The voltmeters are capable of reading microvolts and have high internal resistance. What will each voltmeter read? Make sure you answer is consistent, from every point of view, with Faraday's Law (Eq. 10.3.2).

Solution: Consider the loop containing resistors 1 and 2 with the solenoid passing through it. Choose $\vec{\mathbf{A}}$ in the direction opposite the magnetic field, then the magnetic flux is $\vec{\mathbf{B}}\cdot\vec{\mathbf{A}} = -BA$. The change in the magnetic flux and hence the emf through the loop is given by

$$\varepsilon = -\frac{d\Phi_B}{dt} = +\frac{dB}{dt}A = (10^{-2}\text{ T}\cdot\text{s}^{-1})(20\times 10^{-4}\text{ m}^2) = 2\times 10^{-5}\text{T}\cdot\text{m}^2\cdot\text{s}^{-1} = 20\ \mu\text{V}\,.$$

Therefore the current in the loop is

$$I = \frac{|\varepsilon|}{2R} = \frac{20\ \mu\text{V}}{100\ \Omega} = 0.2\ \mu\text{A}\,.$$

Because the emf is positive, the current is clockwise when looking from right to left in Figure 11.4.2.

Now let's consider the loop that includes resistor 1 and voltmeter 1 that does not enclose magnetic flux. Voltmeter 1 measures the line integral inside the voltmeter from the positive terminal to the negative terminal, hence the meter measures

$$\Delta V_1 = \int_{\substack{+\\ inside\,1}}^{-} \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}\,.$$

If we complete the path from the negative terminal through resistor 1 to the positive terminal then,

$$\int_{\substack{-\\ outside\,1}}^{+} \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = IR_1 = (.2\ \mu\text{A})(50\ \Omega) = 10\ \mu\text{V}\,.$$

Because there is no changing magnetic flux through this loop, Faraday's Law is

$$\int_{\substack{+\\ inside\,1}}^{-} \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} + \int_{\substack{-\\ outside\,1}}^{+} \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = 0\,.$$

Therefore

$$\Delta V_1 + IR_1 = 0 .$$

Hence voltmeter 1 reads

$$\Delta V_1 = -IR_1 = -(.2\ \mu\text{A})(50\ \Omega) = -10\ \mu\text{V} .$$

Note that the higher potential end is at the negative terminal of voltmeter 1.

Now consider Loop 2 that involves resistor 1, voltmeter 2 and encloses changing magnetic flux. Faraday's Law is

$$\underset{inside\ 2}{\int_{+}^{-}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} + \underset{outside\ 1}{\int_{-}^{+}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt} \iint_{loop\ 2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} .$$

The changing magnetic flux is

$$-\frac{d}{dt} \iint_{loop\ 2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = +\frac{dB}{dt} A = 20\ \mu\text{V}$$

Voltmeter 2 measures

$$\Delta V_2 = \underset{inside\ 2}{\int_{+}^{-}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} .$$

We again complete the path from the negative terminal through resistor 1 to the positive terminal, with

$$\underset{ext\ 1}{\int_{-}^{+}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = IR_1 .$$

Therefore Faraday's Law becomes

$$\Delta V_2 + IR_1 = -\frac{d\Phi_B}{dt} .$$

Hence

$$\Delta V_2 = -IR_1 - \frac{d\Phi_B}{dt} = -10\ \mu\text{V} + 20\ \mu\text{V} = 10\ \mu\text{V} .$$

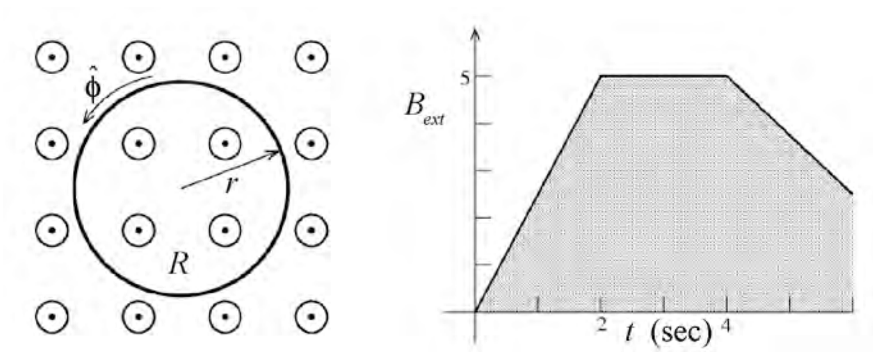
So the higher potential is the positive terminal of voltmeter 2. Both voltmeters are clipped to the same points and the measured voltage difference that have opposite signs,

$$\Delta V_1 = -\Delta V_2 .$$

11.5 How can the Electric Field in an Inductor be Zero?

Students are always confused about the electric field in inductors, in part because of the kinds of problems they have seen. What has changed in our circuit above to make the electric field zero in the wires of the (resistanceless) inductor zero, even though there is a time changing magnetic flux through it? This is a very subtle point and a source of endless confusion, so let's look at it very carefully.

Your intuition that there should be an electric field in the wires of an inductor is based on doing problems like that shown in Figure 11.5.1(a). We have a loop of wire of radius a and total resistance R immersed in an external magnetic field that is out of the plane of the figure and changing in time as shown in Figure 11.5.1(b). In considering this circuit, unlike in our "one-loop" circuit above, we neglect the magnetic field due to the currents in the wire itself, assuming that the external field is much bigger than the self-field. The conclusions we arrive at here can be applied to the self-inductance case as well.



(a) (b)

Figure 11.5.1 (a) Conducting loop in a changing magnetic field. (b) Plot of external magnetic field vs. time.

We calculated the emf in Example 10.3.1 associated with this example and found that

$$\varepsilon = -\frac{dB_{ext}}{dt}\pi a^2 , \tag{11.5.1}$$

and an induced electric field right at the loop given by

$$\vec{\mathbf{E}} = -\frac{1}{2}\frac{dB_{ext}}{dt}a\hat{\boldsymbol{\phi}}, \qquad r < R . \tag{11.5.2}$$

This "induced" electric field is azimuthal and uniformly distributed around the loop as long as the resistance in the loop is uniform.

Thus if the resistance is distributed uniformly around the wire loop, we get a uniform induced electric field in the loop, circulating clockwise for the external magnetic field increasing in time (Eq. (11.5.2)). This electric field causes a current, and the current is directed clockwise in the same sense as the electric field. The total current in the loop will be the total "potential drop" around the loop divided by its resistance R,

$$I = \frac{|\varepsilon|}{R} = \frac{1}{R}\frac{dB_{ext}}{dt}\pi a^2 . \tag{11.5.3}$$

But what happens if we don't distribute the resistance uniformly around the wire loop? For example, let us make the left half of our loop out of wire with resistance R_1 and the right half of the loop out of wire with resistance R_2, with $R = R_1 + R_2$, so that we have the same total resistance as before (Figure 11.5.3). Let us further assume that $R_1 < R_2$. How is the electric field distributed around the loop now?

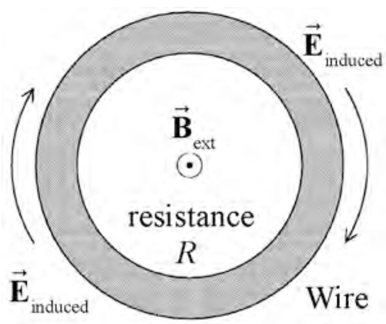


Figure 11.5.2 A loop of wire with resistance R in an external field out of the plane of figure.

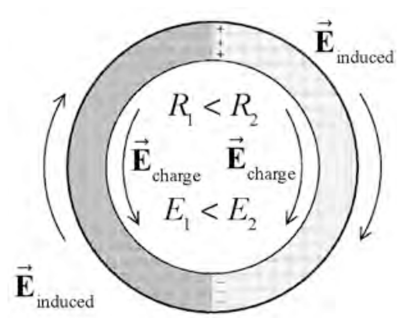


Figure 11.5.3 The electric field in the case of unequal resistances in the loop.

First of all, the electromotive force around the loop (Eq. (11.5.1)) is the same, as is the resistance, so that the current I has to be the same as in Eq. (11.5.3). Moreover it is the same on both sides of the loop by charge conservation. But the electric field in the left half of the loop $\vec{\mathbf{E}}_1$ must now be different from the electric field in the right half of the loop $\vec{\mathbf{E}}_2$. This is so because the line integral of the electric field on the left side is $\pi a E_1$, and from Ohm's Law in macroscopic form, this must be equal to IR_1. Similarly, $\pi a E_2 = IR_2$. Thus

$$\frac{E_1}{E_2} = \frac{R_1}{R_2} \Rightarrow E_1 < E_2 \;\text{ since } R_1 < R_2 \;. \tag{11.5.4}$$

This makes sense. We get the same current on both sides, even though the resistances are different, and we do this by adjusting the electric field on the side with the smaller resistance to ***be*** smaller. Because the resistance is also smaller, we produce the same current as on the opposing side, with this smaller electric field.

But what happened to our uniform electric field. Well there are two ways to produce electric fields—one from time changing currents and their associated time changing magnet fields, and the other from electric charges. Nature accomplishes the reduction of $\vec{\mathbf{E}}_1$ compared to $\vec{\mathbf{E}}_2$ by charging at the junctions separating the two wire segments (Figure 14.4.5), positive on top and negative on bottom.

The total electric field is the sum of the "induced" electric field and the electric field associated with the charges, as shown in the Figure 14.5.3. It is clear that the addition of these two contributions to the electric field will reduce the total electric field on the left (side 1) and enhance it on the right (side 2). The field $\vec{\mathbf{E}}_1$ will always be clockwise, but it can be made arbitrarily small by making $R_1 << R_2$.

Thus we see that we can make a non-uniform electric field in an inductor by using non-uniform resistance, even though our intuition tells us (correctly) that the "induced" electric field should be uniform at a given radius. All that Faraday's Law tells us is that the line integral of the electric field around a closed loop is equal to the negative of the time rate of change of the magnet flux through the enclosed surface. It does not tell us at what locations the electric field is non-zero around the loop, and it may be non-zero (or zero!) in unexpected places. The field in the wire making up the "one-loop" inductor we considered above is zero (or least very small) for exactly the kinds of reasons we have been discussing here.

11.6 Modified Kirchoff's Law (Misleading, see Section 11.4.2)

We now give a modified version of Kirchoff's Law which includes inductors, but you must always be aware that this modified version is wrong (see Section 11.4.2). However it is a useful mnemonic. The modified rule for inductors may be obtained as follows: The polarity of the self-induced emf is such as to oppose the change in current, in accord with Lenz's law. If the rate of change of current is positive, as shown in Figure 11.6.1(a), the self-induced emf ε_L sets up an induced current I_{ind} moving in the opposite direction of the current I to oppose such an increase. The inductor could be replaced by an emf $|\varepsilon_L| = L\,|dI/dt| = +L(dI/dt)$ with the polarity shown in Figure 11.6.1(a). On the other hand, if $dI/dt < 0$, as shown in Figure 11.6.1(b), the induced current I_{ind} set up by the self-induced emf ε_L flows in the same direction as I to oppose such a decrease.

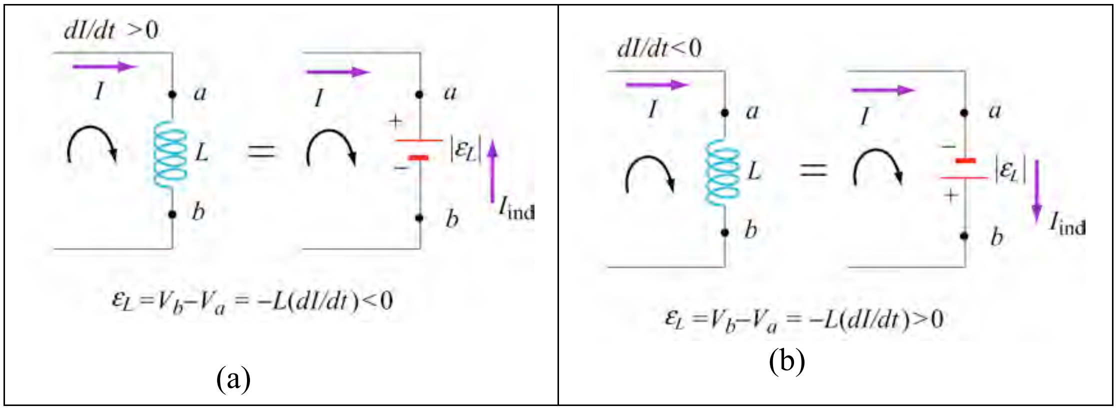


Figure 11.6.1 Modified Kirchhoff's rule for inductors (a) with increasing current, and (b) with decreasing current. See Section 11.4.2 for cautions about the use of this modified rule.

The modified rule for inductors may be obtained as follows: The polarity of the self-induced emf is such as to oppose the change in current, in accord with Lenz's law. If the rate of change of current is positive, as shown in Figure 11.6.1(a), the self-induced emf ε_L sets up an induced current I_{ind} moving in the opposite direction of the current I to oppose such an increase. The inductor could be replaced by an emf $|\varepsilon_L| = L\,|dI/dt| = +L(dI/dt)$ with the polarity shown in Figure 11.6.1(a). On the other hand, if $dI/dt < 0$, as shown in Figure 11.6.1(b), the induced current I_{ind} set up by the self-induced emf ε_L flows in the same direction as I to oppose such a decrease.

We see that whether the rate of change of current in increasing ($dI/dt > 0$) or decreasing ($dI/dt < 0$), in both cases, the change in potential when moving from a to b along the direction of the current I is $V_b - V_a = -L(dI/dt)$. Thus, we have

***Kirchhoff's Loop Rule Modified for Inductors (Misleading, see Section 11.4.2)*:**

If an inductor is traversed in the direction of the current, the "potential change" is $-L(dI/dt)$. On the other hand, if the inductor is traversed in the direction opposite of the current, the "potential change" is $+L(dI/dt)$.

Use of this modified Kirchhoff's rule will give the correct equations for circuit problems that contain inductors. However, keep in mind that it is **misleading** at best, and at some level **wrong** in terms of the physics. Again, we emphasize that Kirchhoff's loop rule was originally based on the fact that the line integral of $\vec{\mathbf{E}}$ around a closed loop was zero. With time-changing magnetic fields, this is no longer so, and thus the sum of the "potential drops" around the circuit, if we take that to mean the negative of the closed loop integral of $\vec{\mathbf{E}}$, is **no longer zero –** in fact it is $+L(dI/dt)$.

11.6.1 Rising Current

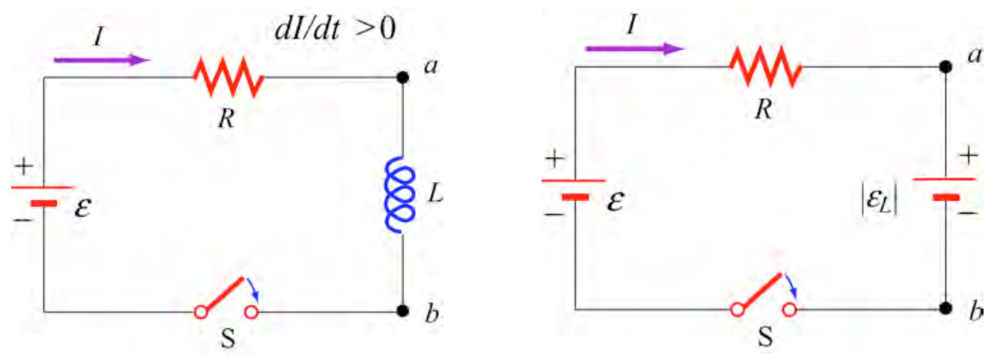


Figure 11.6.2 (a) *RL* Circuit with rising current. (b) Equivalent circuit using the modified Kirchhoff's loop rule.

Consider the RL circuit shown in Figure 11.6.2. At $t = 0$ the switch is closed. We find that the current does not rise immediately to its maximum value ε / R. This is due to the presence of the self-induced emf in the inductor. Using the modified Kirchhoff's rule for increasing current, $dI / dt > 0$, the RL circuit is described by the following differential equation:

$$\varepsilon - IR - |\varepsilon_L| = \varepsilon - IR - L\frac{dI}{dt} = 0 . \qquad (11.6.1)$$

Note that there is an important distinction between an inductor and a resistor. The potential difference across a resistor depends on I, while the potential difference across an inductor depends on dI / dt. The self-induced emf does not oppose the current itself, but the change of current dI / dt. Eq. (11.6.1) can be rewritten as

$$\frac{dI}{I - \varepsilon / R} = -\frac{dt}{L / R} . \qquad (11.6.2)$$

Integrating over both sides and imposing the condition $I(t = 0) = 0$, the solution to the differential equation is

$$I(t) = \frac{\varepsilon}{R}(1 - e^{-t/\tau}) . \qquad (11.6.3)$$

This solution reduces to what we expect for large times, that is $I(\infty) = \varepsilon / R$, but it also shows a continuous rise of the current from $I(t = 0) = 0$ initially to this final value, with a characteristic time τ_L defined by

$$\tau_L = \frac{L}{R} . \qquad (11.6.4)$$

This time constant is known as the inductive time constant. This is the effect of having a non-zero inductance in a circuit, that is, of taking into account the "induced" electric fields that always appear when there are time changing $\vec{\mathbf{B}}$ fields. This is what we expect—the reaction of the system is to try to keep things the same, delaying the build-up of current (or its decay, if we already have current flowing in the circuit).

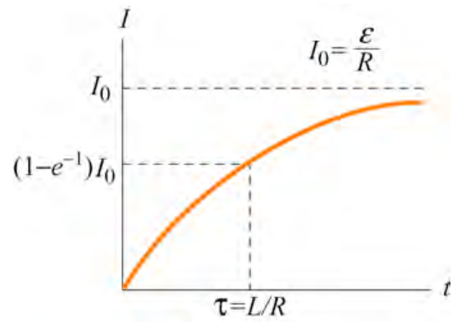


Figure 11.6.3 Current in the RL circuit as a function of time

The qualitative behavior of the current as a function of time is depicted in Figure 11.6.3. Note that after a sufficiently long time, the current reaches its equilibrium value ε / R. The time constant τ is a measure of how fast the equilibrium state is attained; the larger the value of L, the longer it takes to build up the current. A comparison of the behavior of current in a circuit with or without an inductor is shown in Figure 11.6.4. Similarly, the magnitude of the self-induced emf can be obtained as

$$|\varepsilon_L| = \left| -L\frac{dI}{dt} \right| = \varepsilon e^{-t/\tau}, \tag{11.6.5}$$

which is at a maximum when $t = 0$ and vanishes as t approaches infinity. This implies that a sufficiently long time after the switch is closed, self-induction disappears and the inductor simply acts as a conducting wire connecting two parts of the circuit.

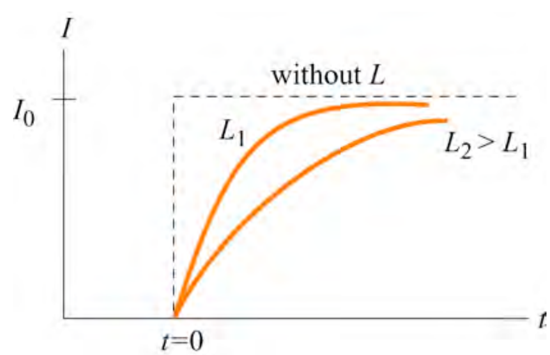


Figure 11.6.4 Behavior of current in a circuit with or without an inductor

To see that energy is conserved in the circuit, we multiply Eq. (11.6.1) by I and obtain

$$I\varepsilon = I^2 R + LI\frac{dI}{dt}. \tag{11.6.6}$$

The left-hand side represents the rate at which the battery delivers energy to the circuit. On the other hand, the first term on the right-hand side is the power dissipated in the resistor in the form of heat, and the second term is the rate at which energy is stored in the inductor. While the energy dissipated through the resistor is irrecoverable, the magnetic energy stored in the inductor can be released later.

11.6.2 Decaying Current

Next we consider the RL circuit shown in Figure 11.6.5. Suppose the switch S_1 has been closed for a long time so that the current is at its equilibrium value ε / R. What happens to the current when at $t = 0$ switches S_1 is opened and S_2 closed?

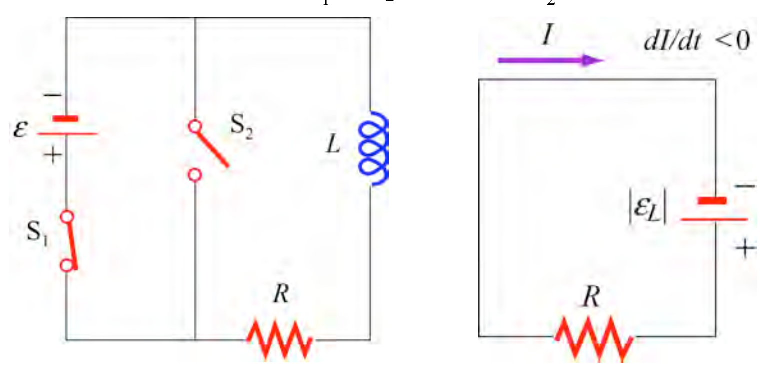


Figure 11.6.5 (a) RL circuit with decaying current, and (b) equivalent circuit.

Applying the modified Kirchhoff's loop rule to the right loop for decreasing current, $dI / dt < 0$, yields

$$|\varepsilon_L| - IR = -L\frac{dI}{dt} - IR = 0, \tag{11.6.7}$$

which can be rewritten as

$$\frac{dI}{I} = -\frac{dt}{L / R}. \tag{11.6.8}$$

The solution to the above differential equation is

$$I(t) = \frac{\varepsilon}{R} e^{-t/\tau}, \tag{11.6.9}$$

where $\tau = L / R$ is the same time constant as in the case of rising current. A plot of the current as a function of time is shown in Figure 11.6.6.

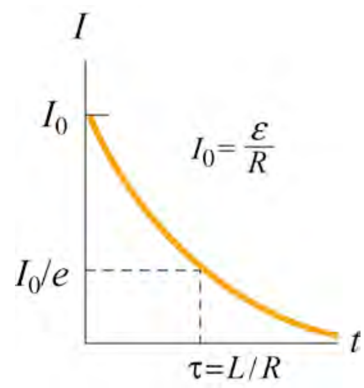


Figure 11.6.6 Decaying current in a RL circuit

11.7 *LC* Oscillations

Consider a LC circuit in which a capacitor is connected to an inductor, as shown in Figure 11.7.1.

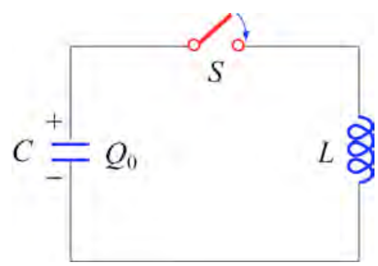


Figure 11.7.1 LC Circuit

Suppose the capacitor initially has charge Q_0. When the switch is closed, the capacitor begins to discharge and the electric energy is decreased. On the other hand, the current created from the discharging process generates magnetic energy that then gets stored in the inductor. In the absence of resistance, the total energy is transformed back and forth between the electric energy in the capacitor and the magnetic energy in the inductor. This phenomenon is called electromagnetic oscillation.

The total energy in the LC circuit at some instant after closing the switch is

$$U = U_C + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 . \qquad (11.7.1)$$

The fact that U remains constant implies that

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0\,. \tag{11.7.2}$$

Eq. (11.7.2) can be rewritten as

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0\,, \tag{11.7.3}$$

where $I = -dQ/dt$ (and $dI/dt = -d^2Q/dt^2$). Notice the sign convention we have adopted here. The negative sign implies that the current I is equal to the rate of *decrease* of charge in the capacitor plate immediately after the switch has been closed. The same equation can be obtained by applying the modified Kirchhoff's loop rule clockwise:

$$\frac{Q}{C} - L\frac{dI}{dt} = 0\,, \tag{11.7.4}$$

followed by our definition of current. The general solution to Eq. (11.5.3) is

$$Q(t) = Q_0\cos(\omega_0 t + \phi)\,, \tag{11.7.5}$$

where Q_0 is the amplitude of the charge, $\omega_0 t + \phi$ is the **phase**, and ϕ is the **phase constant**. The angular frequency ω_0 is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}\,. \tag{11.7.6}$$

The corresponding current in the inductor is

$$I(t) = -\frac{dQ}{dt} = \omega_0 Q_0 \sin(\omega_0 t + \phi) = I_0\sin(\omega_0 t + \phi)\,, \tag{11.7.7}$$

where $I_0 = \omega_0 Q_0$. From the initial conditions $Q(t=0) = Q_0$ and $I(t=0) = 0$, the phase constant ϕ can be determined to be $\phi = 0$. Thus, the solutions for the charge and the current in our *LC* circuit are

$$Q(t) = Q_0\cos(\omega_0 t)\,, \tag{11.7.8}$$

and

$$I(t) = I_0\sin(\omega_0 t)\,. \tag{11.7.9}$$

The time dependence of $Q(t)$ and $I(t)$ are depicted in Figure 11.7.2.

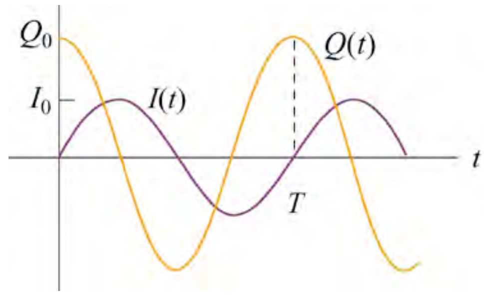


Figure 11.7.2 Charge and current in the LC circuit as a function of time

Using Eqs. (11.7.8) and (11.7.9), we see that at any instant of time, the electric energy and the magnetic energies are given by

$$U_E = \frac{Q^2(t)}{2C} = \left(\frac{Q_0^2}{2C}\right)\cos^2(\omega_0 t), \tag{11.7.10}$$

and

$$\begin{aligned} U_B &= \frac{1}{2}LI^2(t) = \frac{LI_0^2}{2}\sin^2(\omega_0 t) \\ &= \frac{L(-\omega_0 Q_0)^2}{2}\sin^2(\omega_0 t) = \left(\frac{Q_0^2}{2C}\right)\sin^2(\omega_0 t) \end{aligned} \tag{11.7.11}$$

respectively. One can easily show that the total energy remains constant:

$$U = U_E + U_B = \left(\frac{Q_0^2}{2C}\right)\cos^2\omega_0 t + \left(\frac{Q_0^2}{2C}\right)\sin^2\omega_0 t = \frac{Q_0^2}{2C} \tag{11.7.12}$$

The electric and magnetic energy oscillation is illustrated in Figure 11.7.3.

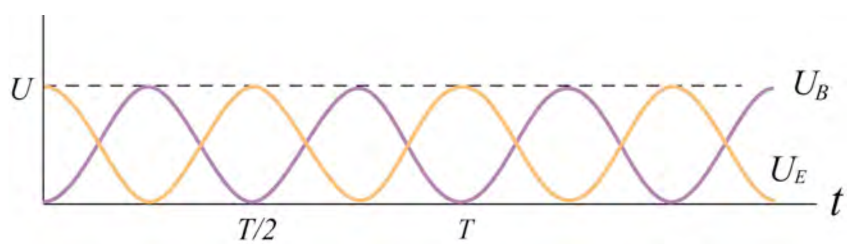


Figure 11.7.3 Electric and magnetic energy oscillations

The mechanical analog of the LC oscillations is the mass-spring system, shown in Figure 11.7.4.

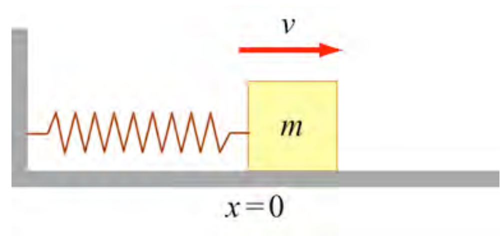


Figure 11.7.4 Mass-spring oscillations

If the mass is moving with a speed v and the spring having a spring constant k is displaced from its equilibrium by x, then the energy E_{mech} of this mechanical system is

$$E_{\text{mech}} = K + U_{\text{sp}} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 , \tag{11.7.13}$$

where K and U_{sp} are the kinetic energy of the mass and the potential energy of the spring, respectively. In the absence of friction, E_{mech} is constant and we obtain

$$\frac{dE_{\text{mech}}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0 . \tag{11.7.14}$$

Using $v = dx/dt$ and $dv/dt = d^2x/dt^2$, the above equation may be rewritten as

$$m\frac{d^2x}{dt^2} + kx = 0 . \tag{11.7.15}$$

The general solution for the displacement is

$$x(t) = A\cos(\omega_0 t + \phi) \tag{11.7.16}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{11.7.17}$$

is the angular frequency and A is the amplitude of the oscillations. Thus, at any instant in time, the energy of the system may be written as

$$\begin{aligned} E_{\text{mech}} &= \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) \\ &= \frac{1}{2}kA^2\left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)\right] = \frac{1}{2}kA^2 \end{aligned} . \tag{11.7.18}$$

11.8 The *RLC* Series Circuit

We now consider a series RLC circuit that contains a resistor, an inductor and a capacitor, as shown in Figure 11.8.1.

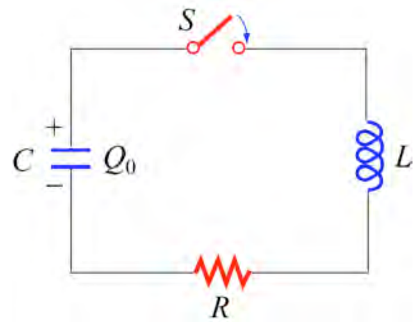


Figure 11.8.1 A series RLC circuit

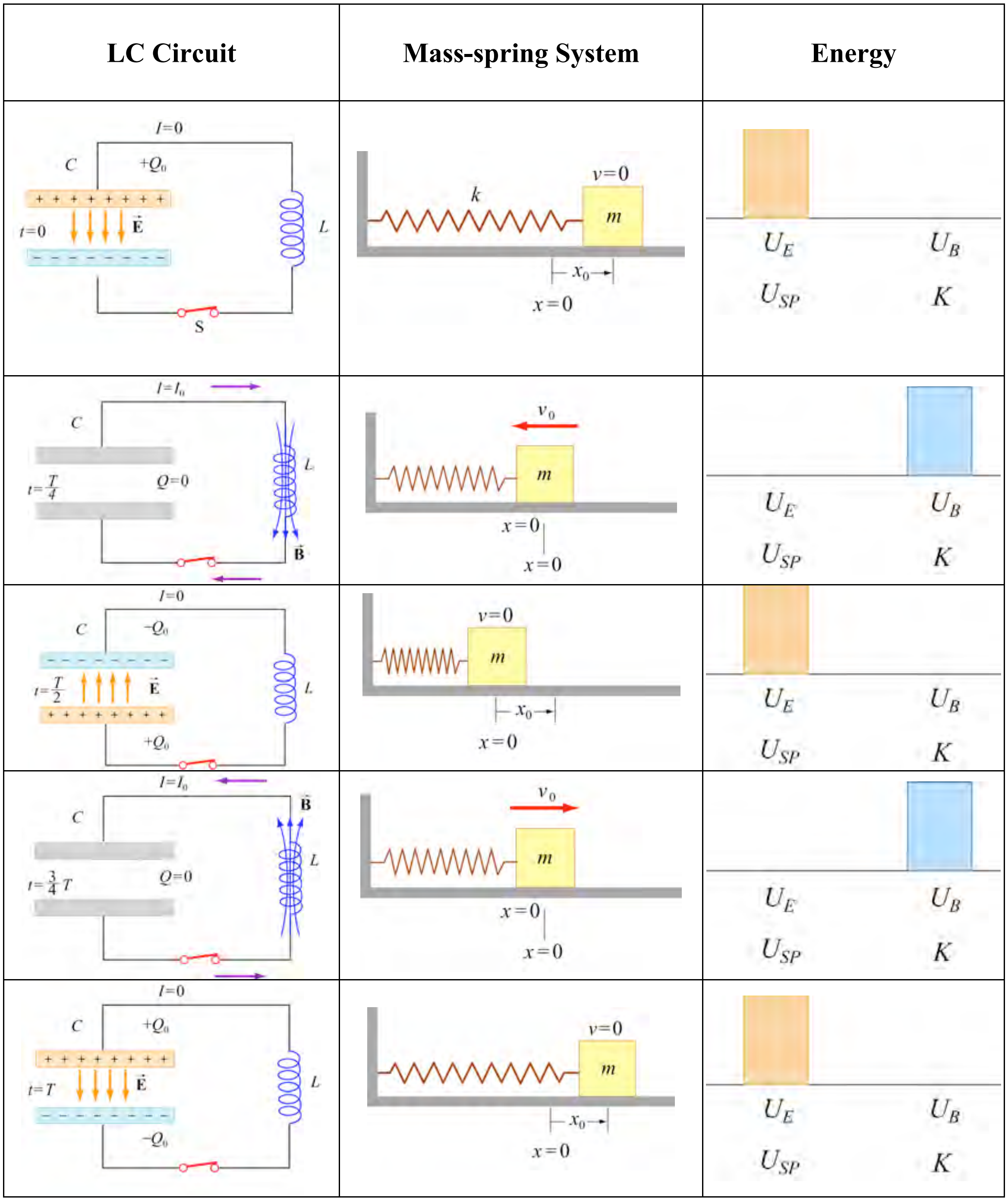


Figure 11.7.5 Energy oscillations in the LC Circuit and the mass-spring system

In Figure 11.7.5 we illustrate the energy oscillations in the LC Circuit and the mass-spring system (harmonic oscillator).

The capacitor is initially charged to Q_0. After the switch is closed current will begin to flow. However, unlike the LC circuit energy will be dissipated through the resistor. The rate at which energy is dissipated is

$$\frac{dU}{dt} = -I^2 R. \tag{11.8.1}$$

where the negative sign on the right-hand side implies that the total energy is decreasing. After substituting Eq. (11.7.2) for the left-hand side of the above equation, we obtain the following differential equation

$$\frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = -I^2 R. \tag{11.8.2}$$

Again, by our sign convention where current is equal to the rate of *decrease* of charge in the capacitor plates, $I = -dQ/dt$. Dividing both sides by I, the above equation can be rewritten as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0. \tag{11.8.3}$$

For small R (the underdamped case, see Appendix 1), one can readily verify that a solution to the above equation is

$$Q(t) = Q_0 e^{-\gamma t}\cos(\omega' t + \phi), \tag{11.8.4}$$

where the damping factor is

$$\gamma = \frac{R}{2L}. \tag{11.8.5}$$

The angular frequency of the damped oscillations

$$\omega' = \sqrt{\omega_0^2 - \gamma^2}. \tag{11.8.6}$$

The constants Q_0 and ϕ are real quantities to be determined from the initial conditions. In the limit where the resistance vanishes, $R = 0$, we recover the undamped natural angular frequency $\omega_0 = 1/\sqrt{LC}$. There are three possible scenarios and the details are discussed in Appendix 1 (Section 11.10).

The mechanical analog of the series RLC circuit is the damped harmonic oscillator system. The equation of motion for this system is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0, \tag{11.8.7}$$

where the velocity-dependent term accounts for the non-conservative, dissipative force

$$F = -b\frac{dx}{dt} \tag{11.8.8}$$

with b being the damping coefficient. The correspondence between the RLC circuit and the mechanical system is summarized in Table 11.8.1. (Note that the sign of the current I depends on the physical situation under consideration.)

	RLC Circuit	Damped Harmonic Oscillator
Variable s	Q	x
Variable ds/dt	$\pm I$	v
Coefficient of s	$1/C$	k
Coefficient of ds/dt	R	b
Coefficient of d^2s/dt^2	L	m
Energy	$LI^2/2$	$mv^2/2$
	$Q^2/2C$	$kx^2/2$

Table 11.8.1 Correspondence between the RLC circuit and the mass-spring system

11.9 Summary

- Using Faraday's law of induction, the **mutual inductance** of two coils is given by

$$M_{12} = \frac{N_{12}\Phi_{12}}{I_1} = M_{21} = \frac{N_1\Phi_{21}}{I_2} = M\,.$$

- The induced emf in coil 2 due to the change in current in coil 1 is given by

$$\varepsilon_2 = -M\frac{dI_1}{dt}\,.$$

- The **self-inductance** of a coil with N turns is

$$L = \frac{N\Phi_B}{I}$$

where Φ_B is the magnetic flux through one turn of the coil.

- The **self-induced emf** responding to a change in current inside a coil current is

$$\varepsilon_L = -L\frac{dI}{dt}.$$

- The inductance of a solenoid with *N* turns, cross sectional area *A* and length *l* is

$$L = \frac{\mu_0 N^2 A}{l}.$$

- If a battery, supplying an emf ε, is connected to an inductor and a resistor in series at time $t = 0$, then the current in this ***RL*** **circuit** as a function of time is

$$I(t) = \frac{\varepsilon}{R}(1 - e^{-t/\tau}).$$

  where $\tau = L / R$ is the time constant of the circuit. If the battery is removed in the RL circuit, the current will decay as

$$I(t) = \frac{\varepsilon}{R}e^{-t/\tau}.$$

- The **magnetic energy** stored in an inductor with current I passing through is

$$U_B = \frac{1}{2}LI^2.$$

- The **magnetic energy density** at a point with magnetic field strength B is

$$u_B = \frac{B^2}{2\mu_0}.$$

- The differential equation for an oscillating ***LC*** **circuit** is

$$\frac{d^2Q}{dt^2} + \omega_0^2 Q = 0,$$

  where $\omega_0 = \dfrac{1}{\sqrt{LC}}$ is the angular frequency of oscillation. The charge on the capacitor as a function of time is given by

$$Q(t) = Q_0\cos(\omega_0 t + \phi)$$

and the current in the circuit is

$$I(t) = -\frac{dQ}{dt} = +\omega_0 Q_0 \sin(\omega_0 t + \phi) .$$

- The total energy in an LC circuit is, using $I_0 = \omega_0 Q_0$,

$$U = U_E + U_B = \frac{Q_0^2}{2C}\cos^2\omega_0 t + \frac{LI_0^2}{2}\sin^2\omega_0 t = \frac{Q_0^2}{2C} .$$

- The differential equation for an ***RLC* circuit** is

$$\frac{d^2Q}{dt^2} + 2\gamma\frac{dQ}{dt} + \omega_0^2 Q = 0 ,$$

where $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and $\gamma = R/2L$. In the underdamped case, the charge on the capacitor as a function of time is

$$Q(t) = Q_0 e^{-\gamma t}\cos(\omega' t + \phi) ,$$

where $\omega' = \sqrt{\omega_0^2 - \gamma^2}$.

11.10 Appendix 1: General Solutions for the *RLC* Series Circuit

In Section 11.8, we have shown that the RLC circuit is characterized by the following differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \tag{11.10.1}$$

whose solutions is given by

$$Q(t) = Q_0 e^{-\gamma t}\cos(\omega' t + \phi) \tag{11.10.2}$$

where the damping factor is

$$\gamma = \frac{R}{2L} \tag{11.10.3}$$

and the angular frequency of the damped oscillations is

$$\omega' = \sqrt{\omega_0^2 - \gamma^2} . \tag{11.10.4}$$

There are three possible scenarios, depending on the relative values of γ and ω_0.

Case 1: Underdamping

When $\omega_0 > \gamma$, or equivalently, ω' is real and positive, the system is said to be *underdamped*. This is the case when the resistance is small. Charge oscillates (the cosine function) with exponentially decaying amplitude $Q_0 e^{-\gamma t}$. However, the frequency of this damped oscillation is less than the undamped oscillation, $\omega' < \omega_0$. The qualitative behavior of the charge on the capacitor as a function of time is shown in Figure 11.10.1.

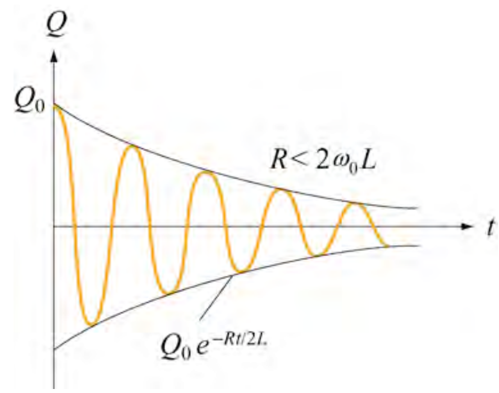


Figure 11.10.1 Underdamped oscillations

As an example, suppose the initial condition is $Q(t=0) = Q_0$. The phase constant is then $\phi = 0$, and

$$Q(t) = Q_0 e^{-\gamma t} \cos(\omega' t). \tag{11.10.5}$$

The corresponding current is

$$I(t) = -\frac{dQ}{dt} = Q_0 \omega' e^{-\gamma t} \left[\sin(\omega' t) + (\gamma / \omega') \cos(\omega' t) \right]. \tag{11.10.6}$$

For small R, the above expression may be approximated as

$$I(t) \approx \frac{Q_0}{\sqrt{LC}} e^{-\gamma t} \sin(\omega' t + \delta) \tag{11.10.7}$$

where

$$\delta = \tan^{-1}\left(\frac{\gamma}{\omega'}\right). \tag{11.10.8}$$

The derivation is left to the readers as an exercise.

Case 2: Overdamping

In the *overdamped* case, $\omega_0 < \gamma$, implying that ω' is imaginary. There is no oscillation in this case. By writing $\omega' = i\beta$, where $\beta = \sqrt{\gamma^2 - \omega_0^2}$, one may show that the most general solution can be written as

$$Q(t) = Q_1 e^{-(\gamma+\beta)t} + Q_2 e^{-(\gamma-\beta)t}, \tag{11.10.9}$$

where the constants Q_1 and Q_2 can be determined from the initial conditions.

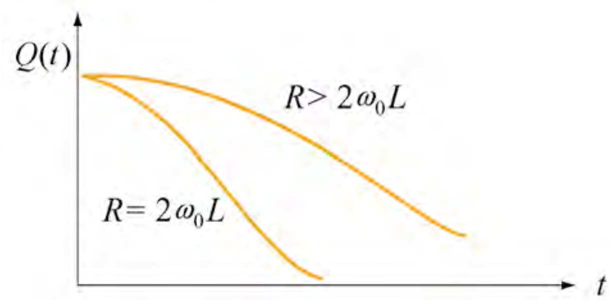


Figure 11.10.2 Overdamping and critical damping

Case 3: Critical damping

When the system is critically damped, $\omega_0 = \gamma$, $\omega' = 0$. Again there is no oscillation. The general solution is

$$Q(t) = (Q_1 + Q_2 t)e^{-\gamma t}, \tag{11.10.10}$$

where Q_1 and Q_2 are constants which can be determined from the initial conditions. In this case one may show that the energy of the system decays most rapidly with time. The qualitative behavior of $Q(t)$ in overdamping and critical damping is depicted in Figure 11.10.2.

11.10.1 Quality Factor

When the resistance is small, the system is underdamped, and the charge oscillates with decaying amplitude $Q_0 e^{-\gamma t}$. The "quality" of this underdamped oscillation is measured by the so-called "quality factor," Q_{qual} (not to be confused with charge.) The larger the value of Q_{qual}, the less the damping and the higher the quality. Mathematically, Q_{qual} is defined as

$$Q_{\text{qual}} = \omega'\left(\frac{\text{energy stored}}{\text{average power dissipated}}\right) = \omega' \frac{U}{|dU/dt|}. \qquad (11.10.11)$$

Using Eq. (11.10.2) the electric energy stored in the capacitor is

$$U_E = \frac{Q(t)^2}{2C} = \frac{Q_0^{\,2}}{2C} e^{-2\gamma t}\cos^2(\omega' t + \phi). \qquad (11.10.12)$$

To obtain the magnetic energy, we approximate the current as

$$\begin{aligned} I(t) &= -\frac{dQ}{dt} = Q_0 \omega' e^{-\gamma t}\left[\sin(\omega' t + \phi) + \left(\frac{\gamma}{\omega'}\right)\cos(\omega' t + \phi)\right] \\ &\approx Q_0 \omega' e^{-\gamma t} \sin(\omega' t + \phi) \qquad (11.10.13) \\ &\approx \frac{Q_0}{\sqrt{LC}} e^{-\gamma t} \sin(\omega' t + \phi) \end{aligned}$$

assuming that $\omega' \gg \gamma$ and $\omega'^2 \approx \omega_0^2 = 1/LC$. Thus, the magnetic energy stored in the inductor is given by

$$.U_B = \frac{1}{2}LI^2 \approx \frac{LQ_0^{\,2}}{2}\omega'^2 e^{-2\gamma t}\sin^2(\omega' t + \phi) \approx \frac{Q_0^{\,2}}{2C} e^{-2\gamma t}\sin^2(\omega' t + \phi). \qquad (11.10.14)$$

Adding up the two terms, the total energy of the system is

$$U = U_E + U_B \approx \frac{Q_0^{\,2}}{2C} e^{-2\gamma t}\cos^2(\omega' t + \phi) + \frac{Q_0^{\,2}}{2C} e^{-2\gamma t}\sin^2(\omega' t + \phi) = \left(\frac{Q_0^{\,2}}{2C}\right) e^{-2\gamma t}. \qquad (11.10.15)$$

Differentiating the expression with respect to t then yields the rate of change of energy

$$\frac{dU}{dt} = -2\gamma\left(\frac{Q_0^{\,2}}{2C} e^{-2\gamma t}\right) = -2\gamma U. \qquad (11.10.16)$$

Thus, the quality factor becomes

$$Q_{\text{qual}} = \omega' \frac{U}{|dU/dt|} = \frac{\omega'}{2\gamma} = \frac{\omega' L}{R}. \qquad (11.10.17)$$

As expected, the smaller the value of R, the greater the value of Q_{qual}, and therefore the higher the quality of oscillation.

11.11 Appendix 2: Stresses Transmitted by Magnetic Fields

In Chapter 9, we showed that the magnetic field due to an infinite sheet in the *xy*-plane carrying a surface current $\vec{\mathbf{K}} = K\hat{\mathbf{i}}$ is given by

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z > 0 \\ \dfrac{\mu_0 K}{2}\hat{\mathbf{j}}, & z < 0 \end{cases} . \tag{11.11.1}$$

Now consider two sheets separated by a distance d carrying surface currents in the opposite directions, as shown in Figure 11.11.1.

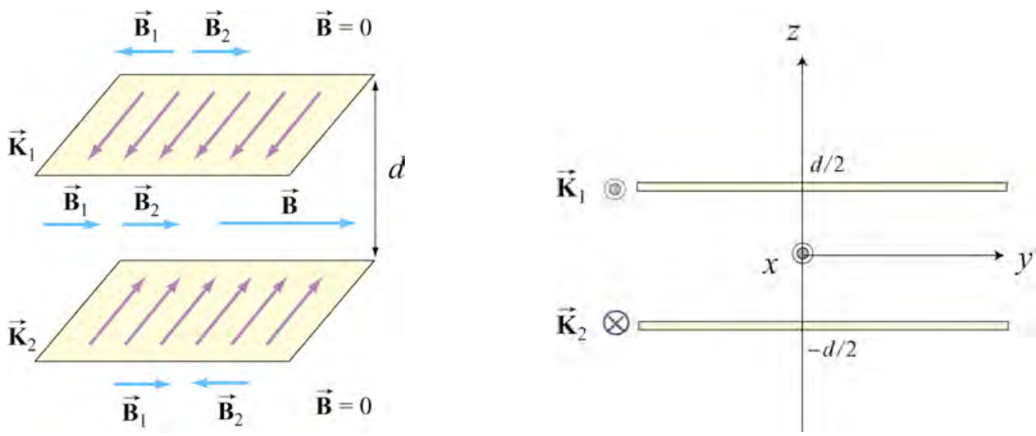


Figure 11.11.1 Magnetic field due to two sheets carrying surface current in the opposite directions

Using the superposition principle, we may show that the magnetic field is non-vanishing only in the region between the two sheets, and is given by

$$\vec{\mathbf{B}} = \mu_0 K\,\hat{\mathbf{j}}, \quad -d/2 < z < d/2 . \tag{11.11.2}$$

Using Eq. (11.3.8) the magnetic energy stored in this system is

$$U_B = \frac{B^2}{2\mu_0}(Ad) = \frac{(\mu_0 K)^2}{2\mu_0}(Ad) = \frac{\mu_0}{2}K^2(Ad) , \tag{11.11.3}$$

where A is the area of the plate. The corresponding magnetic energy density is

$$u_B = \frac{U_B}{Ad} = \frac{\mu_0}{2} K^2 . \qquad (11.11.4)$$

Now consider a small current-carrying element $I d\vec{\mathbf{s}}_1 = (K\,\Delta y)\Delta x\,\hat{\mathbf{i}}$ on the upper plate (Recall that K has dimensions of current/length). The force experienced by this element due to the magnetic field of the lower sheet is

$$d\vec{\mathbf{F}}_{21} = I d\vec{\mathbf{s}}_1 \times \vec{\mathbf{B}}_2 = (K\Delta y\,\Delta x\,\hat{\mathbf{i}}) \times \left(\frac{\mu_0}{2} K\,\hat{\mathbf{j}}\right) = \frac{\mu_0}{2} K^2 (\Delta x \Delta y)\hat{\mathbf{k}} . \qquad (11.11.5)$$

The force points in the $+\hat{\mathbf{k}}$ direction and therefore is repulsive. This is expected since the currents flow in opposite directions. Since $d\vec{\mathbf{F}}_{21}$ is proportional to the area of the current element, we introduce force per unit area, $\vec{\mathbf{f}}_{21}$, and write

$$\vec{\mathbf{f}}_{21} = \vec{\mathbf{K}}_1 \times \vec{\mathbf{B}}_2 = \frac{\mu_0}{2} K^2\,\hat{\mathbf{k}} = u_B\,\hat{\mathbf{k}} , \qquad (11.11.6)$$

using Eq. (11.11.4). The magnitude of the force per unit area, f_{21}, is exactly equal to the magnetic energy density u_B. Physically, f_{21} may be interpreted as the magnetic pressure

$$f_{21} = P = u_B = \frac{B^2}{2\mu_0} \qquad (11.11.7)$$

The repulsive force experienced by the sheets is shown in Figure 11.11.2.

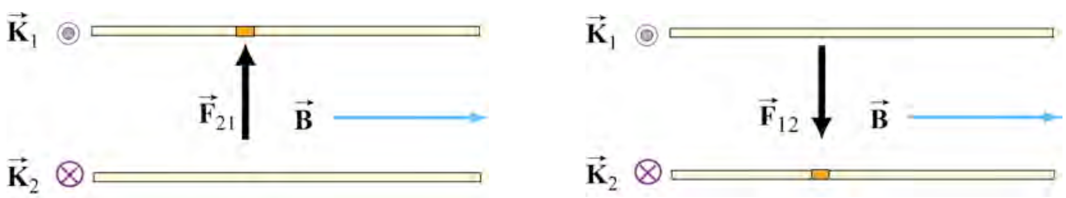


Figure 11.11.2 Magnetic pressure exerted on (a) the upper plate, and (b) the lower plate

11.12 Problem-Solving Strategies

11.12.1 Calculating Self-Inductance

The self-inductance L of an inductor can be calculated using the following steps.

1. Assume a steady current I for the inductor, which may be a conducting loop, a solenoid, a toroid, or coaxial cables.

2. Choose an appropriate cross section S and compute the magnetic flux through S using

$$\Phi_B = \iint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}.$$

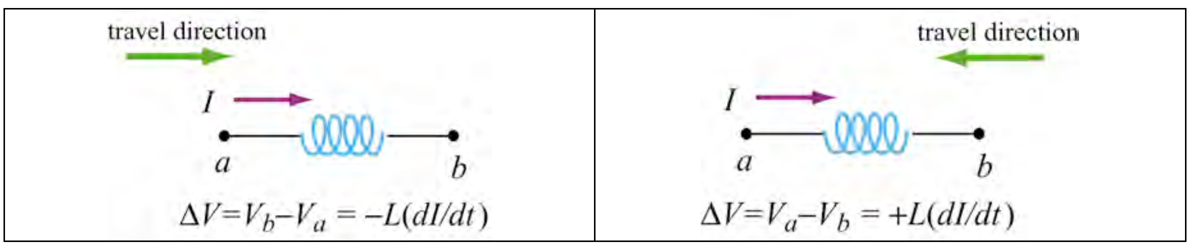
   If the surface were bounded by N turns of wires, then the total magnetic flux through the surface would be $N\Phi_B$.

3. The inductance may be obtained as

$$L = \frac{N\Phi_B}{I}.$$

11.12.2 Circuits containing inductors

Three types of single-loop circuits were examined in this chapter: RL, LC and RLC. To set up the differential equation for a circuit, we apply the Kirchhoff's loop and junction rules, as we did in Chapter 7 for the RC circuits. For circuits that contain inductors, the corresponding modified Kirchhoff's rule is schematically shown below.



Note that the "potential difference" across the inductor is proportional to dI/dt, the rate of change of current. The situation simplifies if we are only interested in the long-term behavior of the circuit where the currents have reached their steady state and $dI/dt = 0$. In this limit, the inductor acts as a short circuit and can simply be replaced by an ideal wire.

11.13 Solved Problems

11.13.1 Energy stored in a toroid

A toroid consists of N turns and has a rectangular cross section, with inner radius a, outer radius b and height h (see Figure 11.2.3). Find the total magnetic energy stored in the toroid.

Solution: In Example 11.3 we showed that the self-inductance of a toroid is

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right).$$

Thus, the magnetic energy stored in the toroid is simply

$$U_B = \frac{1}{2} L I^2 = \frac{\mu_0 N^2 I^2 h}{4\pi} \ln\left(\frac{b}{a}\right). \tag{11.13.1}$$

Alternatively, the energy may be interpreted as being stored in the magnetic field. For a toroid, the magnetic field is (see Chapter 9)

$$B = \frac{\mu_0 N I}{2\pi r}$$

and the corresponding magnetic energy density is

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{\mu_0 N^2 I^2}{8\pi^2 r^2}. \tag{11.13.2}$$

The total energy stored in the magnetic field can be found by integrating over the volume. We choose the differential volume element to be a cylinder with radius r, width dr and height h, so that $dV = 2\pi r h\, dr$. This leads to

$$U_B = \int u_B\, dV = \int_a^b \left(\frac{\mu_0 N^2 I^2}{8\pi^2 r^2}\right) 2\pi r h\, dr = \frac{\mu_0 N^2 I^2 h}{4\pi} \ln\left(\frac{b}{a}\right). \tag{11.13.3}$$

Thus, both methods yield the same result.

11.13.2 Magnetic Energy Density

A wire of nonmagnetic material with radius R and length l carries a current I that is uniformly distributed over its cross-section. What is the magnetic energy inside the wire?

Solution: Applying Ampere's law, the magnetic field at distance $r \le R$ can be obtained as

$$B(2\pi r) = \mu_0 J(\pi r^2) = \mu_0 \left(\frac{I}{\pi R^2}\right)(\pi r^2). \tag{11.13.4}$$

The magnitude of the magnetic field is

$$B = \frac{\mu_0 I r}{2\pi R^2}. \tag{11.13.5}$$

Because the magnetic energy density (energy per unit volume) is given by

$$u_B = \frac{B^2}{2\mu_0}. \tag{11.13.6}$$

The magnetic energy stored in the system becomes

$$U_B = \int_0^R \frac{B^2}{2\mu_0}\left(2\pi r l\, dr\right) = \frac{\mu_0 I^2 l}{4\pi R^4}\int_0^R r^3\, dr = \frac{\mu_0 I^2 l}{4\pi R^4}\left(\frac{R^4}{4}\right) = \frac{\mu_0 I^2 l}{16\pi}. \tag{11.13.7}$$

11.13.3 Mutual Inductance

An infinite straight wire carrying current I is placed to the left of a rectangular loop of wire with width w and length l, as shown in the Figure 11.13.1. Determine the mutual inductance of the system.

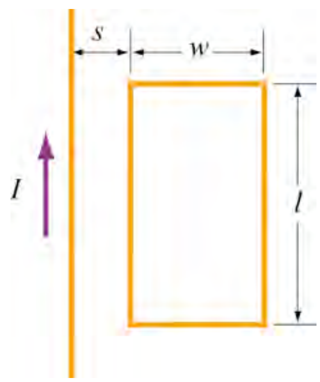


Figure 11.13.1 Rectangular loop placed near long straight current-carrying wire

Solution: To calculate the mutual inductance M, we first need to know the magnetic flux through the rectangular loop. The magnetic field at a distance r away from the straight wire is $B = \mu_0 I / 2\pi r$, using Ampere's law. The total magnetic flux Φ_B through the loop can be obtained by summing over contributions from all differential area elements $dA = l dr$,

$$\Phi_B = \int d\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{\mu_0 I L}{2\pi}\int_s^{s+w}\frac{dr}{r} = \frac{\mu_0 I l}{2\pi}\ln\left(\frac{s+w}{s}\right). \tag{11.13.8}$$

Thus, the mutual inductance is

$$M = \frac{\Phi_B}{I} = \frac{\mu_0 l}{2\pi} \ln\left(\frac{s+w}{s}\right). \qquad (11.13.9)$$

11.13.4 *RL* Circuit

Consider the circuit shown in Figure 11.13.2 below.

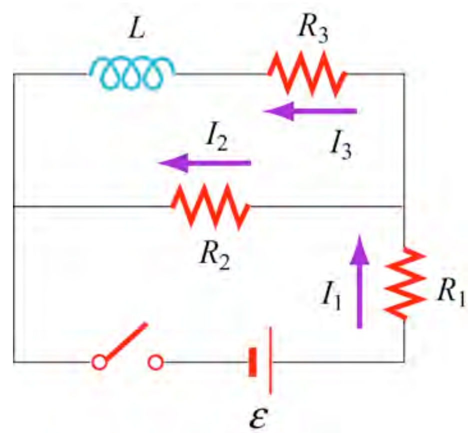


Figure 11.13.2 *RL* circuit

Determine the current through each resistor

(a) immediately after the switch is closed.

(b) a long time after the switch is closed.

Suppose the switch is reopened a long time after it's been closed. What is each current

(c) immediately after it is opened?

(d) after a long time?

Solution:

(a) Immediately after the switch is closed, the current through the inductor is zero because the self-induced emf prevents the current from rising abruptly. Therefore, $I_3 = 0$. Since $I_1 = I_2 + I_3$, we have $I_1 = I_2$.

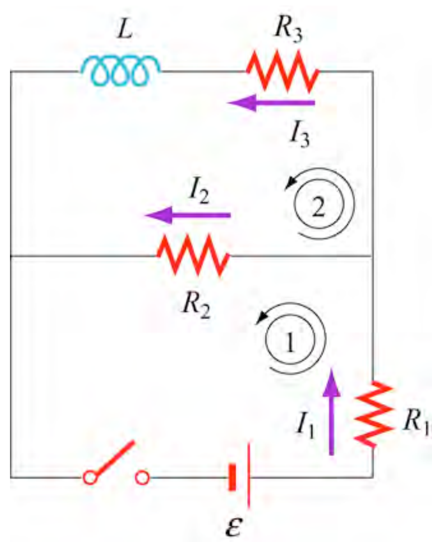


Figure 11.13.3

Applying Kirchhoff's rules to the first loop shown in Figure 11.13.3 yields

$$I_1 = I_2 = \frac{\varepsilon}{R_1 + R_2} . \tag{11.13.10}$$

(b) After the switch has been closed for a long time, there is no induced emf in the inductor and the currents will be constant. Kirchhoff's loop rule gives for the first loop

$$\varepsilon - I_1 R_1 - I_2 R_2 = 0 , \tag{11.13.11}$$

and for the second loop

$$I_2 R_2 - I_3 R_3 = 0 . \tag{11.13.12}$$

Combining the two equations with the junction rule $I_1 = I_2 + I_3$, we obtain

$$\begin{aligned} I_1 &= \frac{(R_2 + R_3)\varepsilon}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ I_2 &= \frac{R_3\,\varepsilon}{R_1 R_2 + R_1 R_3 + R_2 R_3} . \\ I_3 &= \frac{R_2\,\varepsilon}{R_1 R_2 + R_1 R_3 + R_2 R_3} \end{aligned} \tag{11.13.13}$$

(c) Immediately after the switch is opened, the current through R_1 is zero, i.e., $I_1 = 0$. This implies that $I_2 + I_3 = 0$. On the other hand, loop 2 now forms a decaying RL circuit and I_3 starts to decrease. Thus,

$$I_3 = -I_2 = \frac{R_2\varepsilon}{R_1R_2 + R_1R_3 + R_2R_3}. \tag{11.13.14}$$

(d) A long time after the switch has been closed all currents will be zero. That is, $I_1 = I_2 = I_3 = 0$.

11.13.5 *RL* Circuit

In the circuit shown in Figure 11.13.4, suppose the circuit is initially open. At time $t = 0$ it is thrown closed. What is the current in the inductor at a later time t?

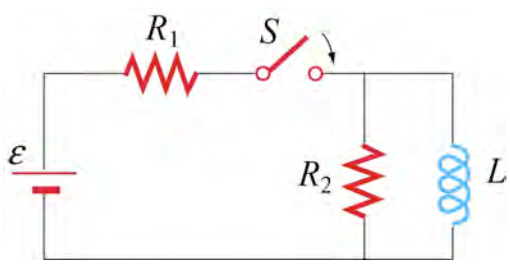


Figure 11.13.4 RL circuit

Solution: Let the currents through R_1, R_2 and L be I_1, I_2 and I, respectively, as shown in Figure 11.13.5.

From Kirchhoff's junction rule, we have $I_1 = I_2 + I$. Similarly, applying Kirchhoff's loop rule to the left loop yields

$$\varepsilon - (I + I_2)R_1 - I_2R_2 = 0. \tag{11.13.15}$$

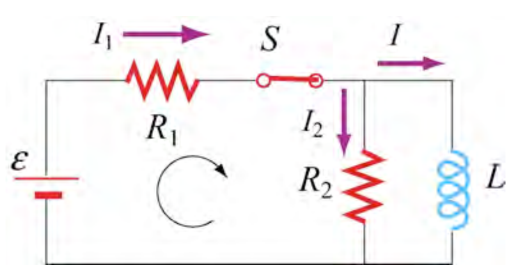


Figure 11.13.5

Similarly, for the outer loop, the modified Kirchhoff's loop rule gives

$$\varepsilon - (I + I_2)R_1 = L\frac{dI}{dt}. \tag{11.13.16}$$

The two equations can be combined to yield

$$I_2 R_2 = L\frac{dI}{dt} \quad \Rightarrow \quad I_2 = \frac{L}{R_2}\frac{dI}{dt}. \tag{11.13.17}$$

Substituting into Eq. (11.13.15) the expression obtained above for I_2, we have

$$\varepsilon - \left(I + \frac{L}{R_2}\frac{dI}{dt}\right)R_1 - L\frac{dI}{dt} = \varepsilon - IR_1 - \left(\frac{R_1 + R_2}{R_2}\right)L\frac{dI}{dt} = 0. \tag{11.13.18}$$

Dividing the equation by $(R_1 + R_2)/R_2$ leads to

$$\varepsilon' - IR' - L\frac{dI}{dt} = 0, \tag{11.13.19}$$

where

$$R' = \frac{R_1 R_2}{R_1 + R_2}, \quad \varepsilon' = \frac{R_2 \varepsilon}{R_1 + R_2}. \tag{11.13.20}$$

The differential equation can be solved and the solution is given by

$$I(t) = \frac{\varepsilon'}{R'}(1 - e^{-R't/L}). \tag{11.13.21}$$

Using the equations in (11.13.20), we have that

$$\frac{\varepsilon'}{R'} = \frac{\varepsilon R_2/(R_1 + R_2)}{R_1 R_2/(R_1 + R_2)} = \frac{\varepsilon}{R_1}. \tag{11.13.22}$$

The current through the inductor may be rewritten as

$$I(t) = \frac{\varepsilon}{R_1}(1 - e^{-R't/L}) = \frac{\varepsilon}{R_1}(1 - e^{-t/\tau}) \tag{11.13.23}$$

where $\tau = L/R'$ is the time constant.

11.13.6 *LC* Circuit

Consider the circuit shown in Figure 11.13.6. Suppose the switch that has been connected to point a for a long time is suddenly thrown to b at $t = 0$.

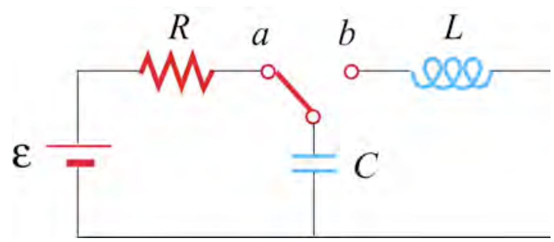


Figure 11.13.6 LC circuit

Find the following quantities:

(a) the frequency of oscillation of the LC circuit.

(b) the maximum charge that appears on the capacitor.

(c) the maximum current in the inductor.

(d) the total energy the circuit possesses at any time t.

Solution:

(a) The angular frequency of oscillation of the LC circuit is given by $\omega = 2\pi f = 1/\sqrt{LC}$. Therefore, the frequency is

$$f = \frac{1}{2\pi\sqrt{LC}}. \tag{11.13.24}$$

(b) The maximum charge stored in the capacitor before the switch is thrown to b is

$$Q = C\varepsilon. \tag{11.13.25}$$

(c) The energy stored in the capacitor before the switch is thrown is

$$U_E = \frac{1}{2}C\varepsilon^2. \tag{11.13.26}$$

On the other hand, the magnetic energy stored in the inductor is

$$U_B = \frac{1}{2} L I^2 . \qquad (11.13.27)$$

Thus, when the current is at its maximum, all the energy originally stored in the capacitor is now in the inductor

$$\frac{1}{2} C \varepsilon^2 = \frac{1}{2} L I_0^2 . \qquad (11.13.28)$$

This implies a maximum current

$$I_0 = \varepsilon \sqrt{\frac{C}{L}} . \qquad (11.13.29)$$

(d) At any time, the total energy in the circuit would be equal to the initial energy that the capacitance stored, that is

$$U = U_E + U_B = \frac{1}{2} C \varepsilon^2 . \qquad (11.13.30)$$

11.14 Conceptual Questions

1. How would you shape a wire of fixed length to obtain the greatest and the smallest inductance?

2. If the wire of a tightly wound solenoid is unwound and made into another tightly wound solenoid with a diameter 3 times that of the original one, by what factor does the inductance change?

3. What analogies can you draw between an ideal solenoid and a parallel-plate capacitor?

4. In the RL circuit show in Figure 11.14.1, can the self-induced emf ever be greater than the emf supplied by the battery?

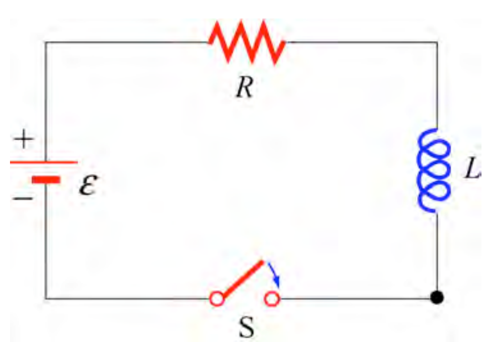


Figure 11.14.1

5. The magnetic energy density $u_B = B^2 / 2\mu_0$ may also be interpreted as the magnetic pressure. Using the magnetic pressure concept, explain the attractive (repulsive) force between two coils carrying currents in the same (opposite) direction.

6. Explain why the LC oscillation continues even after the capacitor has been completely discharged.

7. Explain physically why the time constant $\tau = L / R$ in a RL circuit is proportional to L and inversely proportional to R.

11.15 Additional Problems

11.15.1 Solenoid

A solenoid with a length of 30 cm, a radius of 1.0 cm and 500 turns carries a steady current $I = 2.0\ \text{A}$.

(a) What is the magnetic field at the center of the solenoid along the axis of symmetry?

(b) Find the magnetic flux through the solenoid, assuming the magnetic field to be uniform.

(c) What is the self-inductance of the solenoid?

(d) What is the induced emf in the solenoid if the rate of change of current is $dI / dt = 100\ \text{A/s}$?

11.15.2 Self-Inductance

Suppose you try to wind a wire of length d and radius a into an inductor, which has the shape of a cylinder with a circular cross section of radius r. The windings are tight without wires overlapping. Show that the self-inductance of this inductor is

$$L = \mu_0 \frac{rd}{4a}.$$

11.15.3 Coupled Inductors

(a) If two inductors with inductances L_1 and L_2 are connected in series, show that the equivalent inductance is

$$L_{\text{eq}} = L_1 + L_2 \pm 2M ,$$

where M is their mutual inductance. How is the sign chosen for M? Under what condition can M be ignored?

(b) If the inductors are instead connected in parallel, show that, if their mutual inductance can be ignored, the equivalent inductance is given by

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} .$$

How would you take the effect of M into consideration?

11.15.4 *RL* Circuit

The *RL* circuit shown in Figure 11.15.1 contains a resistor R_1 and an inductance L in series with a battery of emf ε_0. The switch S is initially closed. At $t = 0$, the switch S is opened, so that an additional very large resistance R_2 (with $R_2 >> R_1$) is now in series with the other elements.

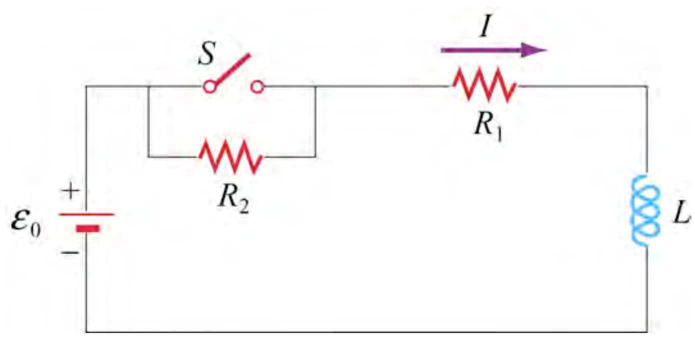


Figure 11.15.1 *RL* circuit

(a) If the switch has been *closed* for a long time before $t = 0$, what is the steady current I_0 in the circuit?

(b) While this current I_0 is flowing, at time $t = 0$, the switch S is opened. Write the differential equation for $I(t)$ that describes the behavior of the circuit at times $t \geq 0$. Solve this equation (by integration) for $I(t)$ under the approximation that $\varepsilon_0 = 0$. (Assume that the battery emf is negligible compared to the total emf around the circuit for times just after the switch is opened.) Express your answer in terms of the initial current I_0, and R_1, R_2, and L.

(c) Using your results from (b), find the value of the total emf around the circuit (which from Faraday's law is $-LdI/dt$) just after the switch is opened. Is your assumption in (b) that ε_0 could be ignored for times just after the switch is opened valid?

(d) What is the magnitude of the potential drop across the resistor R_2 at times $t > 0$, just after the switch is opened? Express your answers in terms of ε_0, R_1, and R_2. How does the potential drop across R_2 just after $t = 0$ compare to the battery emf ε_0, if $R_2 = 100R_1$?

11.15.5 *RL* Circuit

In the circuit shown in Figure 11.15.2, $\varepsilon = 100\ \text{V}$, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 30\ \Omega$, and the inductance L in the right loop of the circuit is 2.0 H. The inductance in the left loop of the circuit is zero.

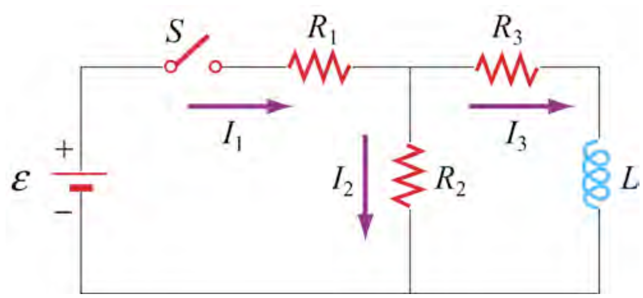


Figure 11.15.2 *RL* circuit

(a) Find I_1 and I_2 immediately after switch S is closed.

(b) Find I_1 and I_2 a long time later. What is the energy stored in the inductor a long time later?

(c) A long, long time later, switch S is opened again. Find I_1 and I_2 immediately after switch S is opened again.

(d) Find I_1 and I_2 a long time after switch S is opened. How much energy is dissipated in resistors R_2 and R_3 between the time immediately after switch S is opened again, and a long time after that?

(e) Give a crude estimate of what "a long time" is in this problem.

11.15.6 Inductance of a Solenoid With and Without Iron Core

(a) A long solenoid consists of N turns of wire, has length l, and cross-sectional area A. Show that the self-inductance can be written as $L = \mu_0 N^2 A / l$. Note that L increases as N^2, and has dimensions of μ_0 times a length (as must always be true).

(b) A solenoid has a length of 126 cm and a diameter of 5.45 cm, with 1870 windings. What is its inductance if its interior is vacuum?

(c) If we now fill the interior with iron with an effective permeability constant $\kappa_m = 968$, what is its inductance?

(d) Suppose we connect this iron core inductor up in series with a battery and resistor, and that the total resistance in the circuit, including that of the battery and inductor, is $10\ \Omega$. How long does it take after the circuit is established for the current to reach 50% of its final value? [Ans. (b) 8.1 mH; (c) 7.88 H; (d) 0.55 s].

11.15.7 *RLC* Circuit

A *RLC* circuit with battery is set up as shown in Figure 11.15.3. There is no current flowing in the circuit until time $t = 0$, when the switch S_1 is closed.

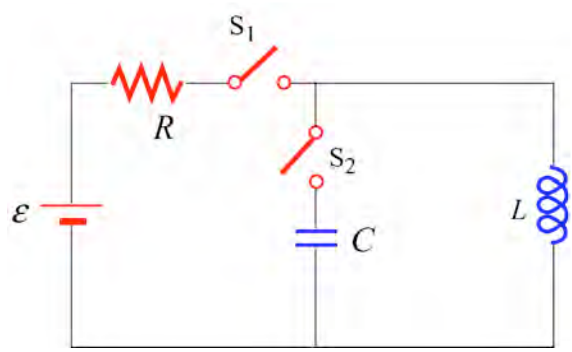


Figure 11.15.3

(a) What is the current I in the circuit at a time $t > 0$ after the switch S_1 is closed?

(b) What is the current I in the circuit a very long time ($t >> L / R$) after the switch S_1 is closed?

(c) How much energy is stored in the magnetic field of the solenoid a very long time $(t >> L/R)$ after the switch is closed?

For the next two questions, assume that a very long time $(t >> L/R)$ after the switch S_1 was closed, the voltage source is disconnected from the circuit by opening the switch S_1 and that the solenoid is simultaneously connected to a capacitor by closing the switch S_2. Assume there is negligible resistance in this new circuit.

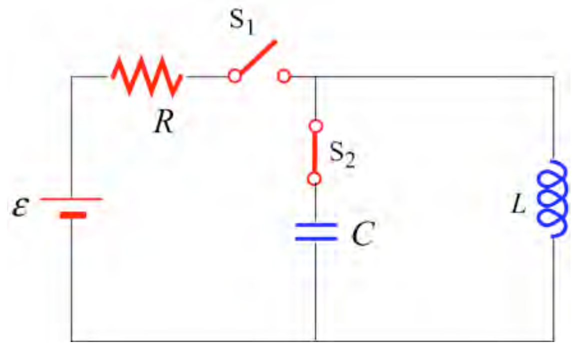


Figure 11.15.4

(d) What is the maximum amount of charge that will appear on the capacitor, in terms of the quantities given?

(e) How long will it take for the capacitor to first reach a maximal charge after the switch S_2 has been closed?

11.15.8 Spinning Cylinder

Two concentric, conducting cylindrical shells are charged up by moving $+Q$ from the outer to the inner conductor, so that the inner conductor has a charge of $+Q$ spread uniformly over its area, and the outer conductor is left with $-Q$ uniformly distributed. The radius of the inner conductor is a; the radius of the outer conductor is b; the length of both is l; and you may assume that $l >> a$ and $l >> b$.

(a) What is the electric field for $r < a$, $a < r < b$, and $r > b$? Give both magnitude and direction.

(b) What is the total amount of energy in the electric field? (Hint: you may use a variety of ways to calculate this, such as using the energy density, or the capacitance, or the potential as a function of Q. It never hurts to check by doing it two different ways.)

(c) If the cylinders are now both spun counterclockwise (looking down the z-axis) at the same angular velocity ω (so that the period of revolution is $T = 2\pi / \omega$), what is the total current (magnitude and sign) carried by each of the cylinders? Give your answer in terms of ω and the quantities from the first paragraph, and consider a current to be positive if it is in the same direction as ω.

(d) What is the magnetic field created when the cylinders are spinning at angular velocity ω? You should give magnitude and direction of $\vec{\mathbf{B}}$ in each of the three regions: $r < a$, $a < r < b$, and $r > b$. (Hint: it's easiest to do this by calculating $\vec{\mathbf{B}}$ from each cylinder independently and then getting the net magnetic field as the vector sum.)

(e) What is the total energy in the magnetic field when the cylinders are spinning at ω?

11.15.9 Spinning Loop

A circular, conducting loop of radius a has resistance R and is spun about its diameter that lies along the y-axis, perpendicular to an external, uniform magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{k}}$. The angle between the normal to the loop and the magnetic field is θ, where $\theta = \omega t$. You may ignore the self-inductance of the loop.

(a) What is the magnetic flux through the loop as a function of time?

(b) What is the emf induced around the loop as a function of time?

(c) What is the current flowing in the loop as a function of time?

(d) At an instant that the normal to the loop aligns with the x-axis, the top of the loop lies on the $+z$-axis. At this moment is the current in this piece of loop in the $+\hat{\mathbf{j}}$ or $-\hat{\mathbf{j}}$ direction?

(e) What is the magnitude of the new magnetic field B_{ind} (as a function of time) created at the center of the loop by the induced current?

(f) Estimate the self-inductance L of the loop, using approximation that the magnetic field strength B_{ind} is uniform over the area of the loop and has the value calculated in part (e).

(g) At what angular speed ω will the maximum induced magnetic field strength B_{ind} equal the external field strength B (therefore thoroughly contradicting the assumption of negligible self-inductance that went into the original calculation of B_{ind})? Express your answer in terms of R and L.

11.15.10 *LC* Circuit

Suppose at $t = 0$ the capacitor in the LC circuit is fully charged to Q_0. At a later time $t = T/6$, where T is the period of the LC oscillation, find the ratio of each of the following quantities to its maximum value:

(a) charge on the capacitor,

(b) energy stored in the capacitor,

(c) current in the inductor, and

(d) energy in the inductor.